Stochastic Observability and Uncertainty Characterization in Simultaneous Receiver and Transmitter Localization

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Abstract—The stochastic observability of simultaneous receiver and transmitter localization is studied. A mobile vehicle-mounted receiver is assumed to draw pseudorange measurements from multiple unknown radio frequency (RF) transmitters and to fuse these measurements through an extended Kalman filter (EKF) to simultaneously localize the receiver and transmitters together with estimating the receiver’s and transmitters’ clock errors. The receiver is assumed to have perfect a priori knowledge of its initial states, while the transmitters’ states are unknown. It is shown that the receiver’s and transmitters’ clock biases are stochastically unobservable and that their estimation error variances will diverge. A lower bound on the divergence rate of the estimation error variances of the receiver’s and transmitters’ clock errors is derived and demonstrated numerically. Simulation and experimental results are presented for an unmanned aerial vehicle (UAV) navigating without GPS signals, using pseudoranges made on unknown terrestrial transmitters. It is demonstrated that despite the receiver’s and transmitters’ clock biases being stochastically unobservable, the EKF produces bounded localization errors.

Index Terms—Stochastic observability, simultaneous localization and mapping, emitter localization, source localization, transmitter mapping.

I. INTRODUCTION

Localizing unknown radio frequency (RF) transmitters is important in domains ranging from identifying rogue transmitters, such as jammers and spoofers [1], [2], to radiolocation via signals of opportunity [3], [4]. Signals of opportunity are ambient RF signals that are not intended as localization or navigation sources, such as AM/FM [5], cellular [6], [7], digital television [8], [9], and iridium [10], [11]. These signals are abundant, diverse in frequency and direction, and received with high carrier-to-noise ratio, making them an attractive alternative to global navigation satellite system (GNSS) signals. However, unlike GNSS space vehicle (SV) states, the states of signals of opportunity transmitters, namely their position and clock error states, may not be known a priori, in which case they must be estimated. These states may be simultaneously estimated alongside the vehicle-mounted receivers’ position, velocity, and clock error states, which provides a self-contained solution that does not require the installation of additional infrastructure [12], [13].

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This estimation problem is referred to as radio simultaneous localization and mapping (SLAM) and is analogous to the SLAM problem in robotics [14]. However, in contrast to the static feature map of the typical SLAM problem which consists of static states (e.g., positions of buildings, walls, poles, trees, etc.), the radio SLAM signal landscape map consists of static states (e.g., positions of terrestrial transmitters) and dynamic stochastic states (e.g., clock bias and drift).

Observability of the SLAM problem in robotics has been extensively studied [15]–[18]. In [19], observability of the radio SLAM problem was thoroughly analyzed through a linearized deterministic observability framework, deriving conditions on the minimal a priori knowledge about the receivers’ and/or transmitters’ states for observability. In [20], a nonlinear deterministic observability framework was utilized to show that receiver-controlled maneuvers reduce the a priori knowledge needed to establish observability. This paper studies the observability of the radio SLAM problem in a stochastic framework to characterize the evolution of the estimation error covariance produced by an extended Kalman filter (EKF) estimating the stochastic dynamic states.

Classic deterministic observability tests do not include the statistics of the: (i) process noise, (ii) measurement noise, or (iii) initial state estimate. The EKF Riccati equation, however, which governs the time evolution of the estimation error covariance, is a function of such statistics. Therefore, a system may pass deterministic observability tests, while there may exist a combination of system statistics for which an EKF would yield estimates with unbounded estimation error variances [21]. For this reason, studying observability via a stochastic framework is of considerable importance to characterize the time evolution of the EKF’s estimation error covariance.

Several stochastic observability notions have been defined in the literature. In [21], [22], a system was said to be stochastically observable if there exists a time such that an estimator could produce a finite estimation error covariance, when no prior information about the system’s state vector is available. In [23], a system was said to be estimable if in estimating its states from measurements, the posterior estimation error covariance matrix is strictly smaller than the prior state covariance matrix. In [24] and [25], the stochastic stability of the discrete-time (DT) and continuous-time EKF were studied and conditions on the initial estimation error and disturbing noise terms were specified that will guarantee bounded estimation error. In [26], stochastic observability (or
estimability) was defined as an assessment of the “degree
of observability.” Thus, in contrast to Boolean deterministic
observability tests, stochastic observability was defined as a
measure to whether an observable system is poorly estimable
due to the gradient vectors comprising the Fisher information
matrix being nearly collinear. In [27], stochastic observability
was used to describe the ability of the estimator to reduce the
entropy of any non-trivial function of its initial state by using
the measurements.

In this paper, the stochastic observability of the radio SLAM
problem is studied by directly analyzing the time evolution of
the estimation error covariance through the Riccati equation.
The radio SLAM problem is found to be stochastically un-
observable when both the receiver’s and transmitters’ clock
biases are simultaneously estimated by showing divergence of
their individual variances. The stochastic observability analysis
in this paper allows for the initial estimation error covariance
to be finite, unlike other existing approaches that assume in-
finite initial uncertainty [21], [22]. This paper makes three con-
tributions. First, a closed-form expression for a lower bound
on the time evolution of the estimation error variances of the
stochastically unobservable states is derived. Second, the lower
bound’s divergence rate is characterized. Third, numerical and
experimental results are presented demonstrating an unmanned
aerial vehicle (UAV)-mounted receiver, navigating in a radio
SLAM fashion by fusing pseudoranges made on unknown
terrestrial signals of opportunity transmitters. It is worth noting
that this paper focuses on a planar environment to simplify the
analysis. Extensions to three-dimensional environments is
expected to follow straightforwardly.

The remainder of the paper is organized as follows. Section
II describes the system dynamics and measurement models.
Section III studies the stochastic observability of the simul-
taneous receiver and transmitter localization problem. Section
IV presents simulation results to validate the findings of Sec-
tion III. Section V provides experimental results. Concluding
remark are given in Section VI.

II. MODEL DESCRIPTION

A. RF Transmitter Dynamics Model

Each RF signal will be assumed to emanate from a spatially-
stationary terrestrial transmitter, and its state vector will consist
of its planar position states \( r_{s,m} \triangleq [x_{s,m}, y_{s,m}]^T \) and clock
error states \( x_{clk,s,m} \triangleq \left[ \delta t_{s,m}, \dot{\delta}t_{s,m} \right]^T \), where \( c \) is the speed
of light, \( \delta t_{s,m} \) and \( \dot{\delta}t_{s,m} \) are the clock bias and drift of the \( m \)th
RF transmitter, respectively, and \( m = 1, \ldots, M \), where \( M \) is the
total number of RF transmitters.

The discretized RF transmitters’ dynamics are given by
\[
x_{s,m}(k+1) = F_s x_{s,m}(k) + w_{s,m}(k), \quad k = 1, 2, \ldots
\]
where
\[
x_{s,m} = \left[ r_{s,m}^T, x_{clk,s,m}^T \right]^T,
F_s = \text{diag} \left[ I_{2\times 2}, F_{clk} \right], \quad F_{clk} = \begin{bmatrix}
1 & T \\
0 & 1
\end{bmatrix},
\]
\( T \) is the constant sampling interval and \( w_{s,m} \) is the process
noise, which is modeled as a DT white noise sequence with
covariance \( Q_{s,m} = \text{diag} \left[ 0_{2\times 2}, c^2 Q_{clk,s,m} \right] \), where
\[
Q_{clk,s,m} = \begin{bmatrix}
S_{\dot{\delta}t_{s,m}} T + S_{\delta t_{s,m}} T^3 & S_{\dot{\delta}t_{s,m}} T^2 \\
S_{\delta t_{s,m}} T^2 & S_{\delta t_{s,m}} T
\end{bmatrix}.
\]
The terms \( S_{\dot{\delta}t_{s,m}} \) and \( S_{\delta t_{s,m}} \) are the clock bias and drift
process noise power spectra, respectively, which can be related
to the power-law coefficients, \( \{ h_{t,s,m} \}_{s,m=2}^{\infty} \), which have been
shown through laboratory experiments to characterize the
power spectral density of the fractional frequency deviation of
an oscillator from nominal frequency according to \( S_{\delta t_{s,m}} \approx \frac{h_{t,s,m}}{2} \) and \( S_{\dot{\delta}t_{s,m}} \approx 2\pi^2 h_{t-2,s,m} \) [28].

B. Receiver Dynamics Model

The receiver’s planar position \( r_r \triangleq [x_r, y_r]^T \) and velocity
\( \dot{r}_r \) will be assumed to evolve according to an arbitrary, but
known, continuous-time dynamics model \( f_{pv} \) (e.g., velocity
random walk or constant turn rate [29]). The receiver’s state
vector \( x_r \) is defined by augmenting the receiver’s position and
velocity states \( x_{pv} \triangleq [r_r^T, \dot{r}_r^T]^T \) with its clock error states,
\( x_{clk,r} \triangleq c \left[ \delta t_r, \dot{\delta}t_r \right]^T \), i.e., \( x_r \triangleq \left[ x_{pv}^T, x_{clk,r}^T \right]^T \). Discretizing
the receiver’s dynamics at a constant sampling period \( T \) yields
\[
x_r(k+1) = f_r[x_r(k)] + w_r(k),
\]
where \( f_r[x_r(k)] \triangleq \left[ f_{pv}^T[x_{pv}(k)] \right] \left[ F_{clk} x_{clk,r}(k) \right]^T \),
\( f_{pv} \) is a vector-valued function, which is obtained by
discretizing \( f_{pv} \) at a constant sampling interval \( T \), \( w_r \)
is the process noise vector, which is modeled as a DT
zero-mean white noise sequence with covariance \( Q_r = \text{diag} \left[ Q_{pv}, c^2 Q_{clk,r} \right] \), where \( Q_{pv} \) is the position and velocity
process noise covariance and \( Q_{clk,r} \) is identical to \( Q_{clk,s,m} \), ex-
cept that \( S_{\dot{\delta}t_{s,m}} \) and \( S_{\delta t_{s,m}} \) are replaced with receiver-
specific spectra, \( S_{\dot{\delta}t_{r}} \) and \( S_{\delta t_{r}} \), respectively.

C. Measurement Model

The pseudorange measurement made by the receiver on the
\( m \)th RF transmitter, after discretization and mild approxima-
tions discussed in [19], is related to the receiver’s and RF
transmitter’s states by
\[
z_{s,m}(k) = \| r_r(k) - r_{s,m} \| + c [\delta t_r(k) - \delta t_{s,m}(k)] + v_{s,m}(k), \tag{1}
\]
where \( \| \cdot \| \) is the Euclidean norm and \( v_{s,m} \) is the measurement
noise, which is modeled as a DT zero-mean white Gaussian
sequence with variance \( \sigma_{s,m}^2 \).

D. Augmented System

The augmented system of an environment comprising one
receiver and \( M \) RF transmitters will be denoted \( \Sigma \) and is given by
\[
\Sigma : \begin{cases}
\dot{x}(k+1) = f[x(k)] + w(k) \\
z(k) = h[x(k)] + v(k)
\end{cases} \tag{2}
\]
where \( f[x(k)] \triangleq \left[ f_r^T[x_r(k)], [\Phi_s x_s(k)]^T \right]^T \); \( x \triangleq [x_r^T, x_s^T]^T \); \( \Phi_s \triangleq \text{diag} \{ \mathbf{P}_1, \ldots, \mathbf{P}_s \}; \( x_s \triangleq [x_{s1}, \ldots, x_{sM}]^T \); \( w \triangleq [w_r^T, w_s^T, \ldots, w_{sM}^T]^T \); and \( v \triangleq [v_{s1}, \ldots, v_{sM}]^T \), with covariance \( \text{cov}(v) \triangleq \mathbf{R} = \text{diag} \{ \sigma_{s1}, \ldots, \sigma_{sM} \} \).

### III. Stochastic Observability Analysis

In this section, an overview of the EKF-based radio SLAM problem is presented, and the system’s stochastic observability is studied according to the definition [21]:

**Definition III.1.** A dynamic system is stochastically observable if and only if there exists a time \( t_b \) such that the estimation error covariance \( \mathbf{P}_\xi(k|k) \) of the state vector \( \xi \) produced by a dynamic estimator remains upper bounded by \( \sigma_b \) in the sense that

\[
\sigma_{\max} \{ \mathbf{P}_\xi(k|k) \} \leq \sigma_b < \infty, \quad \forall kT \geq t_b.
\]

where \( \sigma_{\max} \{ \mathbf{A} \} \) denotes the maximum singular value of \( \mathbf{A} \).

#### A. EKF-based Radio SLAM Overview

The goal of radio SLAM is to provide a receiver to construct and continuously refine a spatiotemporal signal landscape map of the environment, within which the receiver localizes itself in space and time. In the event that GNSS signals become unavailable or untrustworthy, the receiver continues navigating with the aid of this map. In EKF-based radio SLAM, an EKF produces an estimate \( \hat{x}(k|k) \triangleq \mathbb{E}[x(k)|Z^k] \) of \( x(k) \), where \( \mathbb{E}[\cdot | \cdot] \) is the conditional expectation and \( Z^k \) denotes all the measurements up to and including step-time \( k \), i.e., \( Z^k \triangleq \{ z(j) \}_{j=1}^k \). In this paper, it is assumed that the receiver’s initial state vector \( x_r(0) \) is known, which could be obtained from the last instant a reliable GNSS solution was available. The EKF-based radio SLAM prediction (time update) and correction (measurement update) are given by:

1) **Prediction:**
\[
\dot{x}(k+1|k) = \mathbf{F}(k) \hat{x}(k|k), \\
\mathbf{P}_x(k+1|k) = \mathbf{F}(k) \mathbf{P}_x(k|k) \mathbf{F}^T(k) + \mathbf{Q},
\]

2) **Correction:**
\[
\hat{x}(k+1|k+1) = \hat{x}(k+1|k) + \mathbf{L}(k+1) \mathbf{S}^{-1}(k+1) \mathbf{R}^T(k+1) \mathbf{H}(k+1) + \mathbf{Q}, \\
\mathbf{P}_x(k+1|k+1) = \mathbf{P}_x(k+1|k) - \mathbf{L}(k+1) \mathbf{S}^{-1}(k+1) \mathbf{L}^T(k+1),
\]

where \( \mathbf{L}(k+1) \) and \( \mathbf{S}(k+1|k) \) are the prediction error covariance and corrected state estimate, respectively; \( \mathbf{F} \) is the Jacobian of \( f \) evaluated at the current state estimate \( \hat{x}(k|k) \); \( \mathbf{R} \) is the innovation; \( \mathbf{H} \) is the measurement prediction; \( \mathbf{H}(k+1) = \mathbf{H}(k+1|k) \mathbf{L}(k+1) + \mathbf{R} \) is the innovation covariance; and \( \mathbf{H}(k+1) \) is the Jacobian of \( h \) evaluated at \( \hat{x}(k+1|k) \), which has the form

\[
\mathbf{H} = \begin{bmatrix} h_{r,s1}^T & h_{s1}^T & \cdots & 0_{1 \times 4} \\ \vdots & \vdots & \ddots & \vdots \\ h_{r,sM}^T & 0_{1 \times 4} & \cdots & h_{sM}^T \end{bmatrix}.
\]

### B. Stochastically Unobservable Clock Errors

This subsection shows that the EKF estimating the state vector of the system in (2) produces an estimation error covariance matrix \( \mathbf{P}_\xi(k|k) \) whose time evolution grows unboundedly.

Traditional deterministic observability tests provide a necessary, but not sufficient condition for stochastic observability [21]. They also do not incorporate a priori knowledge of the uncertainty about the initial state estimate \( \mathbf{P}_\xi(0|0) \), process noise covariance \( \mathbf{Q} \), or measurement noise covariance \( \mathbf{R} \). Moreover, since they only provide a Boolean assessment of the observability of a system, if the system is stochastically unobservable, they do not yield any characterization of the rate of divergence of unobservable states. In what follows, the time evolution of the Riccati equation is studied to show that the radio SLAM problem is stochastically unobservable and to derive a lower bound for the rate of divergence of stochastically unobservable states.

**Lemma III.1.** If the estimation error covariance matrix \( \mathbf{P}_\xi(k|k) \) is such that

\[
\lim_{k \to \infty} \mathbf{e}_i^T \mathbf{P}_\xi(k|k)e_i = \infty, \quad (3)
\]

where \( \mathbf{e}_i \) denotes the \( i \)th standard basis vector consisting of a 1 in the \( i \)th element and zeros elsewhere, then the \( i \)th state of \( \xi \in \mathbb{R}^n \) is stochastically unobservable, and subsequently the system is stochastically unobservable.

**Proof.** If \( \lim_{k \to \infty} \mathbf{e}_i^T \mathbf{P}_\xi(k|k)e_i = \infty \), then

\[
\lim_{k \to \infty} \text{tr} \{ \mathbf{P}_\xi(k|k) \} = \lim_{k \to \infty} \sum_{i=1}^n \mathbf{e}_i^T \mathbf{P}_\xi(k|k)e_i = \infty,
\]

where \( \text{tr} \{ \mathbf{A} \} \) denotes the trace of \( \mathbf{A} \). From the trace properties, \( \text{tr} \{ \mathbf{P}_\xi(k|k) \} = \sum_{i=1}^n \lambda_i[\mathbf{P}_\xi(k|k)] \), where \( \lambda_i[\mathbf{A}] \) denotes the \( i \)th eigenvalue of \( \mathbf{A} \). Since \( \mathbf{P}_\xi(k|k) \) is symmetric positive definite, its singular values \( \sigma_i[\mathbf{P}_\xi(k|k)] = \lambda_i[\mathbf{P}_\xi(k|k)] \). Therefore

\[
\lim_{k \to \infty} \text{tr} \{ \mathbf{P}_\xi(k|k) \} = \lim_{k \to \infty} \sum_{i=1}^n \lambda_i[\mathbf{P}_\xi(k|k)] = \lim_{k \to \infty} \sum_{i=1}^n \sigma_i[\mathbf{P}_\xi(k|k)] = \infty.
\]

Since \( n \) is finite, then at least the largest singular value will grow unboundedly, i.e., \( \lim_{k \to \infty} \sigma_{\max} \{ \mathbf{P}_\xi(k|k) \} = \infty \), where \( \sigma_{\max} \{ \mathbf{A} \} = \max \{ \sigma_i[\mathbf{A}] \} \), making the system stochastically unobservable.

**Theorem III.1.** The radio SLAM problem consisting of one receiver with knowledge of its initial states and \( M \) unknown...
RF transmitters is stochastically unobservable regardless of the receiver’s motion. Moreover, δt_r and \( \{\delta t_{sm}\}_{m=1}^{M} \) are stochastically unobservable states.

Proof. The proof will proceed in two main steps. First, three simplified systems will be defined, denoted \( \Sigma_1 \), \( \Sigma_{II} \), and \( \Sigma_{III} \), where \( \Sigma_1 \) is a simplified form of \( \Sigma \) and each subsequent system is a simplified version of the preceding one. It is shown that if the subsequent system is stochastically unobservable, then the preceding system must be stochastically unobservable as well. Second, \( \Sigma_{III} \) is shown to be stochastically unobservable according to Definition III.1 by invoking lemma III.1.

Step 1: First, define \( \Sigma_1 \) as a system with (i) known RF transmitter position states \( \{r_{sm}\}_{m=1}^{M} \) and (ii) no process noise driving the receiver’s position and velocity states (i.e., \( Q_{pr} = 0 \), e.g., a receiver moving with a constant velocity. From (i) and (ii), and since \( r_r(0) \) is known, it is obvious that \( \{r_{sm}\}_{m=1}^{M} \) and \( r_r \) are known \( \forall k \) and need not be estimated by the EKF, simplifying the system to be estimated to a linear time-invariant (LTI) system, given by

\[
\Sigma_1: \begin{cases}
    x_{\text{clk}}(k + 1) = \Phi_{\text{clk}} x_{\text{clk}}(k) + w_{\text{clk}}(k) \\
    z_{\text{clk}}(k) = H_{\text{clk}} x_{\text{clk}}(k) + v(k),
\end{cases}
\]

where

\[
\begin{align*}
    x_{\text{clk}} &\triangleq [x_{\text{clk},r}, x_{\text{clk},s_1}, \ldots, x_{\text{clk},s_M}]^T \in \mathbb{R}^{(2+2M)}, \\
    \Phi_{\text{clk}} &\triangleq \text{diag}[F_{\text{clk}}, \ldots, F_{\text{clk}}] \in \mathbb{R}^{[(2+2M) \times (2+2M)]}, \\
    z_{\text{clk}} &\triangleq [z_{\text{clk},s_1}, \ldots, z_{\text{clk},s_M}]^T, \\
    H_{\text{clk}} &= \begin{bmatrix} h_{\text{clk}}^T & -h_{\text{clk}}^T & \cdots & 0_{1 \times 2} \\
    \vdots & \vdots & \ddots & \vdots \\
    h_{\text{clk}}^T & 0_{1 \times 2} & \cdots & -h_{\text{clk}}^T \end{bmatrix},
\end{align*}
\]

where \( w_{\text{clk}} \) is a DT zero-mean white process noise vector with covariance \( Q_{\text{clk}} = \sigma^2\cdot \text{diag}[Q_{\text{clk},r}, Q_{\text{clk},s_1}, \ldots, Q_{\text{clk},s_M}] \). The measurements have the form \( z_{\text{clk},s_m}(k) \triangleq z_m(k) - \|r_r(k) - r_{sm}\| \) for \( m = 1, \ldots, M \).

Since system \( \Sigma_1 \) is LTI, a Kalman filter (KF) may be employed to estimate the state vector \( x_{\text{clk}} \). To incorporate perfect a priori knowledge of \( x_{\text{clk},r}(0) \) in the KF, the corresponding block of the initial estimation error covariance matrix is set to zero. Assuming the initial estimates of \( \{x_{\text{clk},s_m}\}_{m=1}^{M} \) to be uncorrelated, the initial estimation error covariance matrix is given as

\[
1P_{x_{\text{clk}}}(0|0) = \text{diag}[0_{2 \times 2}, 1P_{x_{\text{clk},r}}(0|0)],
\]

where \( 1P_{x_{\text{clk}}}(0|0) \) is the initial estimation error covariance of the KF associated with \( \Sigma_1 \),

\[
1P_{x_{\text{clk},r}}(0|0) \triangleq \text{diag}[1\sigma_1^2, 1\sigma_2^2, \ldots, 1\sigma_2^2, 1\sigma_2^2, 1\sigma_2^2, 0_{1 \times 2}]
\]

is the initial estimation error covariance of the state vectors \( \{x_{\text{clk},s_m}\}_{m=1}^{M} \) and \( 1\sigma_2^2 \) are the variances of the transmitters’ clock bias and drift, respectively.

Define \( \Sigma_{II} \) to be the same as \( \Sigma_1 \) with the additional simplifications that \( S_{\delta t_r} = 0 \) and \( \{S_{\delta t_{sm}}\}_{m=1}^{M} = 0 \). Since \( \Sigma_{II} \) has less process noise than \( \Sigma_1 \), it is obvious that

\[
1P_{x_{\text{clk}}}(k+1|k) \prec \bar{1P}_{x_{\text{clk}}}(k+1|k), \forall k,
\]

where \( \prec \) denotes the difference \( B - A \) being positive definite. Therefore, if \( \Sigma_{II} \) is stochastically unobservable, then \( \Sigma_1 \) must be stochastically unobservable as well.

Define \( \Sigma_{III} \) to be the same as \( \Sigma_{II} \) with the additional simplification that \( R = 0_{M \times M} \). Since \( \Sigma_{III} \) has no measurement noise, it is obvious that \( \bar{1P}_{x_{\text{clk}}}(k+1|k+1) \prec \bar{1P}_{x_{\text{clk}}}(k+1|k+1), \forall k \). Therefore, if \( \Sigma_{III} \) is stochastically unobservable, then \( \Sigma_{II} \) must be stochastically unobservable as well. Also, since for each \( k \) there are \( M \) perfect measurements that are linearly related to \( 2+2M \) states, the state vector order of \( \Sigma_{III} \) may be reduced from \( 2+2M \) to \( 2+M \) and a reduced-order KF may be used. Reduced-order KFs are used in practice to avoid potential numerical issues and reduce computational complexity [30]. It turns out that the reduced-order KF lends itself to a tractable closed-form expression of the time evolution of the associated estimation error covariance; therefore, is used to evaluate the stochastic observability of \( \Sigma_{III} \) for the second part of this proof.

Step 2: An estimate of \( x_{\text{clk}}(k) \) can be computed though

\[
\hat{x}_{\text{clk}}(k|k) = L_1 x_{\text{clk}}(k) + L_2 \hat{x}_{ro}(k|k),
\]

where \( \hat{x}_{ro}(k|k) \) is an estimate produced by a reduced-order KF of the reduced-order state vector \( x_{ro}(k) \triangleq G x_{\text{clk}}(k); (L_1|L_2) \triangleq \begin{bmatrix} H_{\text{clk}} \end{bmatrix}^{-1}; (L_1|L_2) \) is the matrix formed by augmenting the columns of \( L_1 \) and \( L_2 \); \( \begin{bmatrix} H_{\text{clk}} \end{bmatrix}^{-1} \) is the matrix formed by augmenting the rows of \( H_{\text{clk}} \) and \( G \); and \( G \) is the design matrix which is chosen to be

\[
G \equiv \begin{bmatrix} L_2 & 0_{2 \times 2} & \cdots & 0_{2 \times 2} \\
    g^T & -g^T & \cdots & 0_{1 \times 2} \\
    \vdots & \vdots & \ddots & \vdots \\
    g^T & 0_{1 \times 2} & \cdots & -g^T \end{bmatrix}, \quad g \triangleq \begin{bmatrix} 0 \ 1 \end{bmatrix},
\]

so that \( \delta t_r \) is the first state of \( x_{ro} \) and \( \begin{bmatrix} H_{\text{clk}} \end{bmatrix}^{-1} \) is invertible. It is important to note that although the choice of \( G \) and the corresponding reduced-order state vector \( x_{ro} \) are non-unique, the remainder of the proof is invariant to any feasible choice of \( G \) that makes \( \begin{bmatrix} H_{\text{clk}} \end{bmatrix}^{-1} \) invertible.

A reduced-order KF produces \( \hat{x}_{ro}(k+1|k+1) \) and an associated posterior estimation error covariance given by

\[
P_{x_{ro}}(k+1|k+1) = \begin{bmatrix} \Psi - A(k)\Xi P_{x_{ro}}(k|k) \Psi - A(k)\Xi \end{bmatrix}^T + G Q_{\text{clk}} G^T - G Q_{\text{clk}} H_{\text{clk}} A^T(k) - A(k) H_{\text{clk}} Q_{\text{clk}} G^T + A(k) R_{ro} A^T(k),
\]

where \( \Psi \triangleq G \Phi_{\text{clk}} L_2, \Xi \triangleq H_{\text{clk}} \Phi_{\text{clk}} L_2, R_{ro} \triangleq H_{\text{clk}} Q_{\text{clk}} H_{\text{clk}}^T, \) and

\[
A(k) = \begin{bmatrix} \Psi P_{x_{ro}}(k|k) \Xi^T + G Q_{\text{clk}} H_{\text{clk}}^T \end{bmatrix} \cdot \begin{bmatrix} \Xi P_{x_{ro}}(k|k) \Xi^T + R_{ro} \end{bmatrix}^{-1}.
\]

Note that the matrix \( \Xi P_{x_{ro}}(k|k) \Xi^T \) is symmetric positive semi-definite \( \forall k \) and \( R_{ro} \) is symmetric positive definite and time-invariant; therefore, \( \Xi P_{x_{ro}}(k|k) \Xi^T + R_{ro} \) is symmetric positive definite and invertible \( \forall k \).
The estimate of $\hat{x}_{clk}(k+1|k+1)$ is then produced through (5) and its corresponding posterior estimation error covariance is

$$ \begin{align*}
\mathbb{F}_k x_{clk}(k+1|k+1) &= L_2 \mathbb{F}_k x_{clk}(k+1|k+1) L_2^T, \\
\text{(9)}
\end{align*}$$

where $L_2 = [e_1, e_2, e_1 - e_3, \ldots, e_1 - e_{M+2}]^T$. From (9) and the structure of $L_2$, the clock bias estimation error variances of the receiver and the RF transmitters are equal, i.e.,

$$ \begin{align*}
\mathbb{F}_k \sigma_{\delta_{ts}}^2(k|k) &= \{\mathbb{F}_k \sigma_{\delta_{ts}}^2(m|k)|m=1\ldots, M\}. \\
\text{(10)}
\end{align*}$$

This equality holds for any feasible $G$, since there are $M$ perfect measurements; therefore, the biases of the receiver and RF transmitters are linearly related to each other by a deterministic quantity, given by $\delta_{ts} = \delta_{ts} + z_{clk,ts}$, for $m = 1, \ldots, M$. A closed-form expression of the time evolution of $\mathbb{F}_k \sigma_{\delta_{ts}}^2(k|k)$ is found through the following two steps. First, (7) is recursively solved using an initial estimation error covariance given by

$$ \begin{align*}
\mathbb{F}_k \mathbb{P}_x(0) &= G \mathbb{F}_k \mathbb{P}_x(0) G^T, \\
\text{where } \mathbb{F}_k \mathbb{P}_x(0)
\end{align*}$$

has the same structure as (4), except I is replaced with III. Second, the element corresponding to the receiver’s clock bias $e_1^T \mathbb{F}_k \mathbb{P}_x(0) e_1$ is found by substituting the right-hand side of (7) into (9), yielding

$$ \begin{align*}
\mathbb{F}_k \sigma_{\delta_{ts}}^2(k|k) &= \frac{k q_r \prod_{m=1}^M \Omega_m(k)}{\det[k \mathbb{E}_x(0) + R_{ro}]} k = 1, 2, \ldots, \\
\text{(11)}
\end{align*}$$

where $\Omega_m(k) \triangleq q_{sm} + k T^2 \beta_m$, $\beta_m = \mathbb{F}_k \sigma_{\delta_{ts}}^2(0|0)$, $q_r \triangleq c^2 S_{\delta_{ts},T}$, and $q_{sm} \triangleq c^2 S_{\delta_{ts},sm} T$. Finally, to evaluate the limit (3) for the first state of $\mathbb{F}_k \mathbb{P}_x(0)$, the closed-form (11) is used, yielding

$$ \begin{align*}
\lim_{k \to \infty} e_1^T \mathbb{F}_k \mathbb{P}_x(0) e_1 &= \lim_{k \to \infty} k q_r \prod_{m=1}^M \Omega_m(k) \\
\text{lim}_{k \to \infty} k \frac{q_r}{k^{M+1}} &\prod_{m=1}^M (1 + \frac{q_{sm} + T^2 \beta_m}{k \mathbb{E}_x(0) + R_{ro}}) = \infty. \\
\text{(12)}
\end{align*}$$

Therefore, stochastic unobservability follows from Lemma III.1. □

**Theorem III.2. The EKF estimating the receiver’s state simultaneously with the states of $M$ terrestrial transmitters, with a priori knowledge about the receiver’s initial state, for the stochastically unobservable system $\Sigma$, produces corresponding estimation error variances $\sigma_{\delta_{ts}}^2$ and $\{\sigma_{\delta_{ts}}^2|m=1\ldots, M\}$, respectively, whose time evolution is lower-bounded by a diverging sequence with a divergence rate $\gamma(k)$, where $\gamma(k) \propto k^{-\infty}$ $c^2 S_{\delta_{ts},T}$.

**Proof.** From Theorem III.1, system $\Sigma$ is stochastically unobservable and the variances $\sigma_{\delta_{ts}}^2$ and $\{\sigma_{\delta_{ts}}^2|m=1\}$ produced by an EKF will diverge and their time evolutions are lower bounded by (11).

Define the divergence rate of the estimation error variance associated with the $i$th state of the vector $\xi \in \mathbb{R}^n$ as

$$ \gamma(k) = e_i^T [U_{\xi,inc}(k) - U_{\xi,red}(k)] e_i, \\
\text{(13)}$$

where $U_{\xi,inc}(k) \triangleq P_{\xi}(k+1|k) - P_{\xi}(k|k)$ is the uncertainty increase from the EKF prediction step and $U_{\xi,red}(k) \triangleq P_{\xi}(k+1|k) - P_{\xi}(k+1|k+1)$ is the uncertainty reduction from the EKF correction step. Substituting (11) into (13) for the first state of $\mathbb{F}_k \mathbb{P}_x(0)$ gives

$$ \begin{align*}
e_1^T [U_{\xi,inc}(k) - U_{\xi,red}(k)] e_1 \\
= \frac{\mathbb{F}_k \sigma_{\delta_{ts}}^2(k+1|k+1|k+1)}{\mathbb{F}_k \sigma_{\delta_{ts}}^2(k|k),} \\
= \frac{(k+1) q_r \prod_{m=1}^M \Omega_m(k+1)}{\det[k \mathbb{E}_x(0) + R_{ro}]} - k q_r \prod_{m=1}^M \Omega_m(k) \\
\text{det}[k \mathbb{E}_x(0) + R_{ro}] = q_r \prod_{m=1}^M (T^2 \beta_m) - \prod_{m=1}^M (T^2 \beta_m) = q_r. \\
\text{(14)}
\end{align*}$$

Evaluating the limit of (14) yields

$$ \lim_{k \to \infty} e_1^T [U_{\xi,inc}(k) - U_{\xi,red}(k)] e_1 \\
= \frac{(k+1) q_r \prod_{m=1}^M (1 + \frac{q_{sm} + T^2 \beta_m}{k \mathbb{E}_x(0) + R_{ro}})}{\det[k \mathbb{E}_x(0) + R_{ro}]} - k q_r \prod_{m=1}^M (T^2 \beta_m) = q_r. \\
\text{(15)}$$

where $q_r \triangleq c^2 S_{\delta_{ts},T}$.

**Theorem III.2** shows that the radio SLAM problem with a priori knowledge about the receiver’s states is stochastically unobservable, since the estimation uncertainty associated with the clock biases of both the receivers and terrestrial transmitters will diverge. Theorem III.2 establishes a lower bound of this divergence, which in the limit, only depends on the quality of the receiver’s clock, characterized by $S_{\delta_{ts},T}$. The following two sections present numerical and experimental results demonstrating radio SLAM.

**IV. Simulation Results**

In this section, an environment consisting of one UAV-mounted receiver and $M = 5$ RF transmitters is simulated to demonstrate that both the receiver’s clock bias $\delta_{ts}$ and the transmitters’ clock biases $\{\delta_{ts,m}|m=1\ldots, M\}$ are stochastically unobservable, as was shown in Theorem III.1 and to demonstrate that the divergence rate $\gamma(k) \propto k^{-\infty}$ $q_r$, as established in Theorem III.2. To this end, two systems are simulated: (i)
system $\Sigma_{\text{III}}$ to demonstrate the divergence rate $\gamma(k) \xrightarrow{k \to \infty} q_r$ (15) and (ii) the full system $\Sigma$ to demonstrate the divergence of the estimation error variances of the clock biases when the receiver’s position and velocity and the transmitters’ positions are also estimated.

First, an estimate $\hat{x}_{\text{clk}}(k|k)$ of $\Sigma_{\text{III}}$’s state vector was computed through (5), using the design matrix (6) and the estimate $\hat{x}_{ro}(k|k)$, which was produced by a reduced-order KF. The reduced-order KF was initialized according to $\hat{x}_{ro}(0|0) \sim N(0, P_{\text{clk}}(0|0))$, where $P_{\text{clk}}(0|0) = G \Sigma_{\text{III}} P_{\text{clk}}(0|0) G^T$. The simulation settings are tabulated in Table I. The non-zero estimation error trajectories $\hat{x}_{ro} \triangleq x_{ro} - \hat{x}_{ro}$ and their associated $\pm 2\sigma$ bounds are plotted in Figs. 1(a)–(b). The time evolution of $\gamma(k) = e_1^T \left[ U_{\text{clk},\text{Inc}}(k) - U_{\text{clk},\text{Inc}}(0) \right] e_1$ is plotted in Fig. 1(c). The estimation error $\hat{x}_{\text{clk}} \triangleq x_{\text{clk}} - \hat{x}_{\text{clk}}$ was reconstructed from (5) and the associated posterior error covariance $\Sigma_{\text{III}} P_{\text{clk}}$ was computed by substituting the reduced-order KF’s posterior estimation error covariance (7) into (9). The estimation error trajectories and corresponding $\pm 2\sigma$ bounds for $\delta t_r$, $\delta t_s$, and $\delta t_s$ are plotted in Figs. 1(d)–(f).

![Fig. 1. Estimation error trajectories (red) and corresponding $\pm 2\sigma$ bounds (black dashed). Figs. (a) and (b) correspond to a reduced-order KF estimating $x_{ro}$ using settings from Table I, where $x_{ro} \triangleq e_1 x_{ro}$. Fig. (c) illustrates the time evolution of $\gamma(k) = e_1^T \left[ U_{\text{clk},\text{Inc}}(k) - U_{\text{clk},\text{Inc}}(0) \right] e_1$ (black) and the value of its limit $q_r$ (blue dotted), where $C = 4.2241493 \times 10^{-5}$. Figs. (d)–(f) correspond the clock errors of the receiver and transmitter 1, which were reconstructed through (5), and their corresponding $\pm 2\sigma$, which were computed using (7) and (9).](Image)

The following can be concluded from these plots. First, $\Sigma_{\text{III}} \sigma_{\delta t_r}^2(k|k) \triangleq \Sigma_{\text{III}} \sigma_{\delta t_s}^2(k|k) \forall k$, as expected from (10). Second, $\Sigma_{\text{III}} \sigma_{\delta t_r}^2$ and $\Sigma_{\text{III}} \sigma_{\delta t_s}^2$ are diverging, implying $\delta t_r$ and $\delta t_s$ are stochastically unobservable. The same behavior was observed for the variances associated with $\delta t_{s,m} \forall m \geq 1$. Third, Fig. 1(c) illustrates that their divergence converges to a constant, $\gamma(k) \xrightarrow{k \to \infty} q_r$, as established in Theorem III.2.

Next, the full system $\Sigma$ was simulated and an EKF was employed to estimate $x(k)$. The purpose of this simulation is to illustrate that $\delta t_r$ and $\delta t_{s,m} \forall m \geq 1$ are stochastically unobservable in the full system $\Sigma$ and to demonstrate the behavior of the estimation errors of the receiver’s position and velocity and the RF transmitters’ positions, along with their corresponding variances. The receiver moved in a favorable trajectory around the RF transmitters. Specifically, the receiver’s position and velocity states were set to evolve according to a constant turn rate model as described in [29], i.e.,

$$f_{\text{pv}}[x_{\text{pv}}(k)] = \begin{bmatrix} 1 & 0 & \frac{\omega T}{2} & 1 - c(\omega T) \\ 0 & 1 & \frac{1 - c(\omega T)}{\omega T} & \frac{\omega T}{2} \\ 0 & 0 & c(\omega T) & -s(\omega T) \\ 0 & 0 & s(\omega T) & c(\omega T) \end{bmatrix} x_{\text{pv}}(k),$$

where $s(\cdot)$ and $c(\cdot)$ denote $\sin(\cdot)$ and $\cos(\cdot)$, respectively, $\omega$ is a known constant turn rate, and $S_w$ is the process noise power spectral density. This type of open-loop trajectory has been demonstrated to produce better estimates than an open-loop velocity random walk trajectory [13]. The EKF initialization settings and receiver and RF transmitters’ initial states are tabulated in Table II. The environment layout and UAV trajectory is illustrated in Fig. 2. The estimation error trajectories and associated $\pm 2\sigma$ bounds are plotted in Figs. 3(a)–(f) and Fig. 3(g)–(j) for the receiver and RF transmitter 1, respectively.

The following can be concluded from the full system simulation plots in Fig. 3. First, while the variance of $\delta t_{s,1}$ decreases, at some point in time, it begins to diverge unboundedly. On the other hand, the variance of $\delta t_r$ starts from zero (due to the prior knowledge about the receiver’s clock bias) and diverges unboundedly with time. Second, although the errors $\delta t_r$ and $\delta t_{s,1}$ are relatively small, their variances will continue to increase and cause the estimation error covariance matrix to become ill-conditioned. Third, despite the stochasti-
Fig. 2. Simulated environment consisting of $M = 5$ RF transmitters (Tx) (orange) and one UAV-mounted receiver traversing a circular orbit (black).

Firstly unobservable clock biases, the estimation error variances appear to have a finite bound for $\hat{x}, \hat{y}, \hat{\delta}_r, \hat{\delta}_s$, $\hat{x}_m$, $\hat{y}_s$, and $\hat{c\delta}_s$. Similar behavior was noted for the estimates associated with the other four RF transmitters.

V. EXPERIMENTAL DEMONSTRATION

A field experiment was conducted in Riverside, California using a UAV to demonstrate the stochastically unobservable clock biases of both a UAV-mounted receiver and multiple cellular transmitters when an EKF-based radio SLAM framework is employed.

To this end, a UAV was equipped with a two-channel Ettus® E312 universal software radio peripheral (USRP). Two antennas were mounted to the UAV and connected to the USRP: (i) a consumer-grade patch GPS antenna and (ii) a consumer-grade omni-directional cellular antenna. The USRP was tuned to (i) 1575.42 MHz to sample GPS L1 C/A signals and (ii) 882.75 MHz to sample cellular signals which were modulated through code division multiple access (CDMA) and were transmitted from nearby cellular towers. The E312 fed the sampled data to the Multichannel Adaptive TRansceiver Information eXtractor (MATRIX) software-defined receiver (SDR) [31], [32], which produced pseudorange observables to all available GPS SVs and to four cellular towers of the U.S. cellular provider Verizon. The pseudoranges were fed to an EKF-based radio SLAM framework that used the cellular pseudoranges to simultaneously estimate the UAV’s and the four unknown transmitters’ states. The experimental setup is illustrated in Fig. 4.

The UAV traversed a commanded trajectory for 130 seconds. The traversed trajectory was obtained from the UAV’s onboard integrated navigation system, which used a GPS, an inertial navigation system (INS), and other sensors. The UAV’s
trajectory was also estimated via the radio SLAM framework described in this paper. The UAV’s and cellular towers’ heights were assumed to be known for the entire duration of the experiment; therefore, this is a two-dimensional radio SLAM problem which is consistent with the stochastic observability analysis conducted in Section III. The EKF-based radio SLAM filter was initialized with a state estimate given by
\[
\hat{x}(0) = [\hat{x}_r(0), \hat{x}_m(0), \ldots, \hat{x}_s(0)]^T
\]
and a corresponding estimation error covariance
\[
P(0) = \text{diag}[P_r(0), P_s_1(0), \ldots, P_s_4(0)].
\]
The UAV-mounted receiver’s initial estimate \(\hat{x}_r(0)\) was set to be perfectly known, i.e., \(P_r(0) \equiv 0_{6 \times 6}\). The transmitters’ initial state estimates were drawn according to \(\tilde{x}_m(0) \sim N([r_{m}, c_{m}]^T, P_m(0))\). The transmitters’ positions \(\{r_{m}\}_{m=1}^4\) were surveyed beforehand according to the framework described in [33] and verified using Google Earth. The initial clock bias and drift
\[
x_{clk,m}(0) = c \left[\delta t_{m}(0), \delta t_{m}(0)\right]^T, \quad m = 1, \ldots, 4,
\]
were solved for by using the initial set of cellular transmitter pseudoranges (1) according to
\[
c\delta t_{m}(0) = \|r_r(0) - r_m\| + c\delta t_{r_1}(0) - z_{m}(0),
\]
\[
c\delta t_{s}(1) = (c\delta t_{s}(1) - c\delta t_{s}(0))/T,
\]
where \(c\delta t_{m}(1) = \|r_r(1) - r_m\| + c\delta t_{r_1}(1) - z_{m}(1)\). The initial uncertainty associated with the transmitters’ states was set to \(P_m(0) \equiv 10^3 \cdot \text{diag}[1, 1, 3, 0.3]\) for \(m = 1, \ldots, 4\).

The process noise covariance of the receiver’s clock \(Q_{clk,r}\) was set to correspond to a typical temperature-compensated crystal oscillator (TCXO) with \(h_{0,r} = 9.4 \times 10^{-20}\) and \(h_{-2,r} = 3.8 \times 10^{-21}\). The process noise covariances of the cellular transmitters’ clocks were set to correspond to a typical oven-controlled crystal oscillator (OCXO) with \(h_{0,m} = 8 \times 10^{-20}\) and \(h_{-2,m} = 4 \times 10^{-23}\), which is usually the case for cellular transmitters [34], [35]. The UAV’s position and velocity states were assumed to evolve according to velocity random walk dynamics with
\[
f_{pv}[x_{pv}(k)] = \begin{bmatrix} I_{2 \times 2} & TI_{2 \times 2} \\ 0_{2 \times 2} & I_{2 \times 2} \end{bmatrix} x_{pv}(k),
\]
\[
Q_{pv} = \begin{bmatrix} T^3S_{pv} & T^2S_{pv} \\ TS_{pv} & 2S_{pv} \end{bmatrix},
\]
where \(T = 0.0267\) s and \(S_{pv} = \text{diag}[0.02, 0.2]\) is the process noise power spectral density matrix, whose value was found empirically. The measurement noise variances \(\{\sigma^2_m\}_{m=1}^4\) were computed beforehand according to the method described in [33], and were found to be \(\sigma^2_1 = 0.7, \sigma^2_2 = 0.2, \sigma^2_3 = 0.7,\) and \(\sigma^2_4 = 0.1\). The trajectory produced by the UAV’s onboard integrated GPS-INS and the one estimated by the radio SLAM framework are plotted in Fig. 5 along with the initial uncertainty ellipses of the 4 transmitters and the final east-north 99\textsuperscript{th} percentile estimation uncertainty ellipses for tower 1. Similar reduction in the final uncertainty ellipses corresponding to the 3 other towers was noted.

The root mean squared error (RMSE) of the UAV’s estimated trajectory was 9.5 meters and the final error was 7.9 meters. These errors were computed with respect to the GPS-INS trajectory. The resulting estimation errors and corresponding \(\pm 2\sigma\) bounds of the vehicle’s east and north position and the \(\pm 2\sigma\) bounds of the clock bias of both the receiver and tower 1 are plotted in Fig. 6. Only the \(\pm 2\sigma\) bounds are shown for the clock biases of both the receiver and tower 1, since the true biases are unknown; therefore, the estimation error trajectories cannot be plotted. Note that while the estimation error variances of the UAV’s east and north position remained bounded, the estimation error variances of the receiver and tower 1 grew unboundedly, indicating their stochastic unobservability, which is consistent with the simulation results presented in Section IV and Theorem III.1.

![Fig. 5. Environment layout and experimental results showing the estimated UAV trajectories from (i) its onboard GPS-INS integrated navigation system (white) and (ii) radio SLAM (green), the initial position uncertainty of each unknown tower, and tower 1 final position estimate and corresponding uncertainty ellipse. Image: Google Earth.](image)

VI. CONCLUSION

The stochastic observability of a simultaneous receiver localization and transmitter mapping problem was studied. It was demonstrated that the system is stochastically unobservable when the clock biases of both a receiver and unknown transmitters are simultaneously estimated and that their associated estimation error variances will diverge. The divergence rate of a sequence lower-bounding the diverging variances was derived and shown to reach a steady-state that only depends on the receiver’s clock quality. Despite the stochastically unobservable clock biases, simulation and experimental results
demonstrated bounded localization errors of a UAV navigating via radio SLAM for 130 seconds without GPS.

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