A Quadratically Constrained Fractional Quadratic Programming Approach for Dilution of Precision Minimization

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Abstract—An approach to find the global optimal solution of the dilution of precision (DOP) problem is presented. The DOP optimization problem considered assumes an environment comprising multiple randomly pre-deployed sensors (or navigation sources) and an additional sensor is to be introduced at the location that minimizes variations of the DOP problem (e.g., weighted geometric DOP (WGDOP), horizontal DOP (HDOP), vertical DOP (VDOP), etc.). It is shown that the DOP problem can be formulated as quadratically constrained fractional quadratic program. An algorithm for solving this program is presented an Monte Carlo simulation results are given demonstrating convergence of the proposed approach to the global optimal solution. Additionally, Monte Carlo simulation results are presented, demonstrating the superiority of the proposed algorithm to nonlinear numerical optimization solvers that often converge to local optima.

Index Terms—Dilution of precision, DOP, sensor placement, navigation, tracking, quadratically constrained fractional quadratic optimization.

I. INTRODUCTION

Optimal sensor placement is crucial in many application domains including source localization, tracking, and navigation [1]–[3]. In source localization and target tracking applications, one is interested in optimally placing the sensor, which makes observations to an unknown source (e.g., emitter) or target, minimizing the estimation error uncertainty about the state of the sensor or target [4], [5]. In navigation applications, one is interested in optimally placing the sensor (e.g., receiver), which makes observations to known sources (e.g., global navigation satellite system (GNSS) satellites or signal of opportunity (SOP) transmitters), minimizing the estimator error uncertainty about the sensor’s state [6], [7]. Several metrics have been defined as cost functions to be optimized, most notably (1) minimizing the geometric dilution of precision (GDOP) or more generally the weighted GDOP (WGDOP), which is equivalent to minimizing the trace of the estimation error covariance matrix [8]–[12] and (2) maximizing the determinant of the GDOP or WGDOP matrix, which is equivalent to maximizing the determinant of the Fisher information matrix [13]. However, all the aforementioned optimization problems are nonconvex, necessitating the use of numerical general-purpose optimization solvers, which tend to be computationally intensive and could converge to a local optimum.

Instead of directly optimizing a functional of the WGDOP matrix, alternative approximating metrics were proposed, such as maximizing the area of the polygon whose vertices are the endpoints of the unit line-of-sight (LOS) vectors from the source to the sensor [14]. In [15], it was shown that this criterion was piecewise concave for the problem of placing an additional sensor to a set of pre-deployed sensors localizing a single source using pseudorange measurements, and a closed-form expression for the optimal two-dimensional (2-D) position of the additional sensor was derived. The problem was generalized to the case of localizing multiple sources and it was shown that optimizing the product of areas yielded a set of parallelizable convex programs [16].

This paper considers the following problem. A number of sensors (navigation sources) are randomly pre-deployed in a three-dimensional (3-D) environment. Where should an additional sensor (navigation source) be deployed to minimize the WGDOP or elements within the WGDOP matrix, e.g., weighted horizontal dilution of precision (WHDOP), weighted vertical dilution of precision (WVDOP), and weighted time dilution of precision (WTDOP)? In contrast to other approximating metrics, this paper considers optimizing the WGDOP directly. It is shown that the WGDOP minimization problem can be formulated as a quadratically constrained fractional quadratic program, to which numerical solutions yielding the global optimum have been developed in the nonvex optimization literature [17].

The remainder of this paper is organized as follows. Section II presents two motivating problems considered by this paper. Section III formulates the sensor placement problem and describes the models employed in the paper. Section IV proposes a method for solving the DOP minimization problem. Section V presents simulation results validating the proposed approach. Concluding remarks are given in Section VI.

II. MOTIVATING PROBLEMS

This papers addresses two equivalent problems. The first problem, illustrated in Fig. 1(a), considers a number of sensors (e.g., receivers) that are pre-deployed in some random configuration, which are localizing a stationary source (e.g., emitter) by making pseudorange observations to this source. A central estimator is used to fuse pseudoranges from all
In such environment, where should the UAV position
definite matrix. In the case of the navigation sources being
which is modeled as a zero-mean Gaussian random vector with
be correlated; hence,

\[ R \]

Subsequently, the estimation error covariance matrix of a
weighted nonlinear least-squares (WNLS) estimator with \( N \)
sensors (navigation sources), denoted \( P_N \), is given by

\[ P_N \triangleq (H_N R_N^{-1} H_N)^{-1}. \]

B. Problem Formulation

The problem addressed in this paper is the optimal placement
of an additional sensor (navigation source) to a set of \( N - 1 \geq 4 \)
pre-deployed sensors (navigation sources) in order to optimize a functional of the localization (navigation solution) estimation error covariance. To this end, two cost functions are defined

\[ g(P_N) \triangleq \text{tr}[TP_N T^T], \]
\[ g'(P_N) \triangleq \text{det}[T' P_N T'^T], \]

where \( \text{tr}[\cdot] \) is the matrix trace, \( \text{det}[\cdot] \) is the matrix determinant, \( T \) is an arbitrary \( L \times 4 \) matrix and \( L \) is a positive integer, and \( T' \) is an arbitrary \( L' \times 4 \) matrix with rank \( L' \) and \( L' > L < 4 \). These conditions on \( T' \) ensure that \( g'(P_N) \) is nonzero. Note that \( H_N = [H_{N-1} \ h_N] \) and \( \{r_{r_j}\}_{j=1}^{N-1} \) and \( r_s \) are fixed. Therefore, \( H_{N-1} \) is constant. The problem is to find \( r_{r_N} \) that minimizes \( g(P_N) \) or \( g'(P_N) \). The vector \( h_N \) may be expressed as \( h_N = [x^T - 1]^T \) where \( x = |r_{rN} - r_s|/r_s \) is the unit LOS vector from the source (UAV) to the \( N \)th sensor (navigation source). One can parameterize \( x \) in terms of the elevation angle \( \theta \) and the azimuth angle \( \phi \) according to

\[ x = \left[ \cos \theta \cos \phi \cos \theta \sin \phi \sin \theta \right]^T, \]

where \( -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \) is the elevation angle and \( 0 \leq \phi < 2\pi \) is the azimuth angle. It can be seen that any sensor (navigation source) position on the ray whose direction is given by \( x \) yields the same estimation error covariance. Subsequently, the problem boils down to finding the vector \( x \) on the unit sphere that minimizes \( g(P_N) \) or \( g'(P_N) \), given by the following the optimization problems

\[ \min_{x \mid x = 1} \ g(P_N) = \text{tr}[TP_N T^T] , \]
\[ \min_{x \mid x = 1} \ g'(P_N) = \text{det}[T' P_N T'^T]. \]

In the rest of the paper, these two problems are generally referred to as the DOP minimization problem. In order to visualize \( g(P_N) \) and \( g'(P_N) \), the following two motivating examples are considered.

In the first example, 4 sensors are randomly placed on the
unit sphere, which was gridded by uniformly sampling the domain of \( \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \) and \( \phi \in [0, 2\pi] \). Next, \( g(P_N) \) and \( g'(P_N) \) were evaluated for \( T = T' = I \) at each \((\theta, \phi)\) pair

[Fig. 1. Two motivating examples: (a) Placing an additional sensor for optimal source localization. (b) Solving for relative SOP position to minimize VDOP.]

III. MODEL DESCRIPTION AND PROBLEM FORMULATION

In this section, the models adopted in the paper are described and the DOP minimization problem is subsequently formulated.

A. Model Description

The state to be estimated is \( \eta \triangleq [r_s^T, c \delta t_s]^T \), which is composed of the 3-D position vector \( r_s \triangleq [x_s, y_s, z_s]^T \) of the source (UAV) and its clock bias \( c \delta t_s \) expressed in meters, where \( c \) is the speed of light and \( \delta t_s \) is the source’s (UAV’s) clock bias expressed in seconds. The position vector of the \( j \)th sensor (navigation source) is given by \( r_{r_j} \triangleq [x_{r_j}, y_{r_j}, z_{r_j}]^T \) and its clock bias by \( c \delta t_{r_j} \), \( j = 1, \ldots, N \), where \( N \geq 5 \) is the total number of sensors (navigation sources). It is assumed that the positions and clock biases of all the sensors (navigation sources) are known at any time-step. Moreover, each sensor is making a pseudorange measurement to the source. Alternatively, the UAV is making pseudorange measurements to each navigation source. The pseudorange measurements may be expressed as

\[ z_j = \| r_{r_j} - r_s \|_2 + c \cdot [\delta t_{r_j} - \delta t_s] + v_j, \]

where \( v \triangleq [v_1, \ldots, v_N]^T \) is the measurement noise vector, which is modeled as a zero-mean Gaussian random vector with covariance \( R_N \) [21]. Note that the measurement noise may be correlated; hence, \( R_N \) is an arbitrary symmetric positive-definite matrix. In the case of the navigation sources being GNSS satellites, it is assumed that \( z_j \) has been corrected for ionospheric and tropospheric delays. The Jacobian matrix \( H_N \) of the measurement vector \( z \triangleq [z_1, \ldots, z_N]^T \) is given by

\[ H_N = [h_1 \ \ldots \ h_N]^T, \]

where

\[ h_j \triangleq \left[ \frac{r_{r_j}^T - r_s^T}{\| r_{r_j} - r_s \|_2}, -1 \right]^T. \]

The second problem, illustrated in Fig. 1(b), considers an unmanned aerial vehicle (UAV) that is navigating via GNSS signals, but whose navigation solution suffers from a large vertical dilution of precision (VDOP). This problem is inherent to GNSS-based navigation, due to the geometric configuration of GNSS satellites being above the UAV. It has been demonstrated that utilizing terrestrial SOP transmitters significantly reduce the VDOP, since now the elevation angle from which the signals are received spans \(-90^\circ \) to \(+90^\circ \) [18]–[20]. In such environment, where should the UAV position itself in order to minimize its VDOP?
and were plotted in two ways as shown in Figs. 2 and 3: (a) as a 3-D pattern plot where \( g(P_N) \) and \( g'(P_N) \) are proportional to the radial distance to the 3-D surface and the corresponding sensor location is its projection onto the unit sphere. The dark blue markers indicate the endpoints of the unit LOS vectors to 4 pre-deployed sensors. The red marker indicates the endpoint of the vector \( x \) that minimizes \( g(P_N) \). (b) Surface plot showing \( g(P_N) \) as a function of the azimuth angle \( \phi \) and elevation angle \( \theta \).

In the second example, a UAV is assumed to have access to 4 GNSS satellites. The satellite elevation mask was set to 10°, i.e., satellites below such elevation mask are not used to produce the navigation solution (this is common in GNSS-based navigation to avoid severely attenuated GNSS signals and multipath). The UAV is trying to solve for its relative position to a terrestrial SOP in order to minimize its VDOP, given by \( g(P_N) = \text{tr} [e_3^T P_N e_3] = e_3^T P_N e_3 \), where \( e_3 \triangleq [0, 0, 1, 0]^T \). Note that the distance to the GNSS satellites is significantly large such that the unit LOS vectors form the UAV to the satellites do not change while the UAV is positioning itself with respect to the SOP. Hence, this problem becomes equivalent to the sensor problem whose solution is the relative position of the SOP with respect to the UAV. This scenario is illustrated in Fig. 4, where the dark blue markers indicate the endpoints of the unit LOS vectors to the 4 GNSS satellites and the red mark indicates the relative position of the SOP that minimizes the VDOP. Since \( e_3^T P_N e_3 \) is a scalar, then \( g(P_N) = g'(P_N) \). Subsequently, only \( g(P_N) \) is plotted. It can be seen from Fig. 4 that \( g(P_N) \) (and consequently \( g'(P_N) \)) is nonconvex and has several local minima and maxima.

In the next section, a method for obtaining the global minimum of \( g(P_N) \) is developed.

IV. DOP MINIMIZATION

In this section, the minimization problems (1) and (2) are formulated as quadratically constrained fractional quadratic programs and the global solutions are subsequently discussed.

A. DOP Minimization as a Quadratically Constrained Fractional Quadratic Program

Let the measurement noise covariance \( R_N \) after placing the \( N \)th sensor (navigation source) have the following partitioning

\[
R_N = \begin{bmatrix} R_{N-1} & r_N \\ r_N^T & \sigma_N^2 \end{bmatrix}.
\]

Let its inverse \( Y_N \) be partitioned according to

\[
Y_N \triangleq R_N^{-1} = \begin{bmatrix} Y_{N-1} & y_N \\ y_N^T & \mu_N \end{bmatrix}.
\]

The estimation error covariance matrix after placing the \( N \)th sensor (navigation source) may then be expressed as

\[
P_N \triangleq \left( H_N^T R_N^{-1} H_N \right)^{-1} = \left( H_N^T R_{N-1} H_N + \mu_N h_N h_N^T \right)^{-1} + h_N y_N^T H_{N-1} + y_N H_{N-1}^T h_N^{-1} = (M + uu^T)^{-1},
\]

where \( M \triangleq H_{N-1}^T Y_{N-1} Y_{N-1}^T H_{N-1} \) and \( u \triangleq \mu_N h_N + \frac{1}{\mu_N} H_{N-1}^T y_N \). It is trivially assumed that \( H_{N-1} \) is

Fig. 2. Visualization of \( g(P_N) = \text{tr} [P_N] \). (a) 3-D pattern plot where \( g(P_N) \) is proportional to the radial distance to the 3-D surface and the corresponding sensor location is its projection onto the unit sphere. The dark blue markers indicate the endpoints of the unit LOS vectors to 4 pre-deployed sensors. The red marker indicates the endpoint of the vector \( x \) that minimizes \( g(P_N) \). (b) Surface plot showing \( g(P_N) \) as a function of the azimuth angle \( \phi \) and elevation angle \( \theta \).

Fig. 3. Visualization of \( g'(P_N) = \text{det} [P_N] \). (a) 3-D pattern plot where \( g'(P_N) \) is proportional to the radial distance to the 3-D surface and the corresponding sensor location is its projection onto the unit sphere. The dark blue markers indicate the endpoints of the unit LOS vectors to 4 pre-deployed sensors. The red marker indicates the endpoint of the vector \( x \) that minimizes \( g'(P_N) \). (b) Surface plot showing \( g'(P_N) \) as a function of the azimuth angle \( \phi \) and elevation angle \( \theta \).

Fig. 4. Visualization of \( g(P_N) = e_3^T P_N e_3 \). (a) 3-D pattern plot where the VDOP \( g(P_N) \) is proportional to the radial distance to the 3-D surface and the corresponding sensor location is its projection onto the unit sphere. The dark blue markers indicate the endpoints of the unit LOS vectors to 4 GNSS satellites. The red marker indicates the endpoint of the vector \( x \) that minimizes the VDOP. (b) Surface plot showing the VDOP as a function of the azimuth angle \( \phi \) and elevation angle \( \theta \).
full column rank. Note that $Y_N$ is positive definite. Using the Schur complement condition for positive definiteness, the following holds

$$Y_{N-1} \succ 0 \quad \text{and} \quad Y_{N-1} - \frac{1}{\mu_N} y_N y_N^T \succ 0,$$

and since $H_{N-1}$ is full column rank, then $M$ is positive definite and so is its inverse.

Using the matrix inversion lemma, $P_N$ may be expressed as

$$P_N = M^{-1} - \frac{M^{-1} u u^T M^{-1}}{1 + u^T M^{-1} u}.$$  \hfill (4)

Next, $g(P_N)$ and $g'(P_N)$ are re-expressed as fractional quadratic cost functions.

1) Trace Minimization: Using (4), $g(P_N)$ may be expressed as

$$g(P_N) = C - \frac{\text{tr} \left[ T M^{-1} u u^T M^{-1} T^T \right]}{1 + u^T M^{-1} u},$$

where $C \triangleq \text{tr} [TM^{-1} T^T]$. Using the cyclic properties of the trace, the cost function may be expressed as

$$g(P_N) = C + \frac{u^T Q u}{1 + u^T M^{-1} u}.$$  \hfill (5)

where $Q \triangleq -M^{-1} T^T T M^{-1}$. Note that $h_N = \left[ x^T, -1 \right]^T$ and let $Q, M^{-1}, \zeta \triangleq Q H_{N-1}^T y_N$, and $\psi \triangleq M^{-1} H_{N-1}^T y_N$ have the following partitioning

$$Q = \begin{bmatrix} A_1 & \bar{b}_1 \\ \bar{b}_1 & \bar{c}_1 \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} A_2 & \bar{b}_2 \\ \bar{b}_2 & \bar{c}_2 \end{bmatrix}, \quad \zeta = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}, \quad \psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}.$$

Then, $g(P_N)$ may be expressed as

$$g(P_N) = g(x) \triangleq C + \frac{g_1(x)}{g_2(x)},$$

where $g_n(x) \triangleq x^T A_n x - 2 \bar{b}_n^T x + c_n$, $n = 1, 2$, and

$$A_1 \triangleq \mu_N^2 A_1, \quad \bar{b}_1 \triangleq \mu_N^2 \bar{b}_1 - \zeta_1, \quad c_1 \triangleq \mu_N^2 \zeta_1 + \frac{1}{\mu_N} y_N^T H_{N-1} Q H_{N-1}^T y_N - 2 \zeta_2,$$

$$A_2 \triangleq \mu_N^2 A_2, \quad \bar{b}_2 \triangleq \mu_N^2 \bar{b}_2 - \psi_1, \quad c_2 \triangleq \mu_N^2 \psi_1 + \frac{1}{\mu_N} y_N^T H_{N-1} M^{-1} H_{N-1}^T y_N - 2 \psi_2 + 1.$$

2) Determinant Minimization: Using (4) and Sylvester’s determinant theorem, $g'(P_N)$ may be expressed as

$$g'(P_N) = C' \left( 1 + \frac{u^T Q u}{1 + u^T M^{-1} u} \right),$$  \hfill (7)

where $Q' \triangleq -C' M^{-1} T^T \left( T' M^{-1} T^T \right)^{-1} T' M^{-1}$ and $C' \triangleq \det \left[ T' M^{-1} T^T \right]$. Let $Q'$ and $\zeta' \triangleq Q H_{N-1}^T y_N$ have the following partitioning

$$Q' = \begin{bmatrix} \bar{A}'_1 & \bar{b}'_1 \\ \bar{b}'_1 & \bar{c}'_1 \end{bmatrix}, \quad \zeta' = \begin{bmatrix} \zeta_1' \\ \zeta_2' \end{bmatrix}.$$

Then, $g'(P_N)$ may be expressed as

$$g'(P_N) = g'(x) \triangleq C' + \frac{g'_1(x)}{g'_2(x)},$$

where $g'_n(x) \triangleq x^T A'_n x - 2 \bar{b}'_n^T x + c'_n$, $n = 1, 2$, and

$$A'_1 \triangleq \mu_N^2 A'_1, \quad \bar{b}'_1 \triangleq \mu_N^2 \bar{b}'_1 - \zeta_1', \quad c'_1 \triangleq \mu_N^2 \zeta_1' + \frac{1}{\mu_N} y_N^T H_{N-1} Q' H_{N-1}^T y_N - 2 \zeta_2',$$

$$A'_2 \triangleq \mu_N^2 A'_2, \quad \bar{b}'_2 \triangleq \mu_N^2 \bar{b}'_2 - \psi_1, \quad c'_2 \triangleq \mu_N^2 \psi_1 + \frac{1}{\mu_N} y_N^T H_{N-1} M^{-1} H_{N-1}^T y_N - 2 \psi_2 + 1.$$

It can be seen that $g_2(x) = g'_2(x)$. Subsequently, the DOP minimization problem in (2) may be posed as

$$\text{minimize } x \in F \; \frac{g'(x)}{g_2(x)}.$$  \hfill (8)

Note that (6) and (8) are of the form

$$\text{minimize } x \in F \; f(x) = C_0 + \frac{f_1(x)}{f_2(x)},$$

where $f(x), f_1(x), f_2(x)$, and $C_0$ can be either $g(x), g_1(x)$, $g_2(x)$, and $C$; or $g'(x), g'_1(x), g'_2(x)$, and $C'$, respectively.

B. Domain Approximation

Although (9) minimizes the ratio of two quadratic forms, it is not a quadratically constrained fractional quadratic program due to the feasible domain. As discussed in [17], the constraint was shown to take the form

$$C_0^2 \leq x^T G x \leq C_2^2,$$  \hfill (10)

where $C_2^2 > C_0^2 \geq 0$ and $G$ is a positive definite matrix. In what follows, a method for transforming the constraint in (9) into the form of (10) is presented.

First, it must be established that $f(x)$ is continuous. It can be seen from (5) and (7) that both denominators are greater than one since $M^{-1} > 0$, i.e., $g_2(x) \geq 1$ and $g'_2(x) \geq 1$. Consequently $f(x)$ is continuous. Next, denote $x_0^*$ the true optimal solution of (9). Since $f(x)$ is continuous, then for every $\delta_0 > 0$ where $\|x - x_0^*\|_2 < \delta_0$, there exists $\epsilon_0 > 0$ such that $|f(x) - f(x_0^*)| < \epsilon_0$. Now, let
\( \epsilon_0 \equiv \epsilon^* < \min_{x \in \mathcal{L}} |f(x) - f(x_0^*)| \), where \( \mathcal{L} \) is the set of local minima on \( \mathcal{F} \) excluding \( x_0^* \). Therefore, there exists \( \delta^* \) such that \( \|x - x_0^*\|^2 < \delta^* \). In order to satisfy the form of the constraint given in (10), the set \( \mathcal{F} \) is approximated with
\[
\mathcal{F}_2 = \{ x \in \mathbb{R}^3 : 1 \leq x^T x \leq 1 + \delta \},
\] (11)
where \( \delta > 0 \) is made infinitely small. This approximation is needed to formulate the DOP minimization problem as the quadratically constrained fractional quadratic problem discussed in the next subsection. It can be seen that
\[
\lim_{\delta \to 0} \mathcal{F}_2 = \mathcal{F}.
\]
Denote \( x_\delta^* \) the solution to
\[
\min_{x \in \mathcal{F}_2} f(x).
\]
Since \( \mathcal{F}_2 \) is not strictly the unit sphere, then \( x_\delta^* \) may not be a unit vector. Define the unit vector \( x^* \) to be a vector along \( x_\delta^* \) as
\[
x^* \triangleq \frac{x_\delta^*}{\|x_\delta^*\|_2}.
\] (12)
This vector \( x^* \) will be shown to converge to the optimal solution of (9) \( x_0^* \). Let \( \delta \) be small enough such that \( \|x_0^* - x_\delta^*\|_2 < \delta^* \). Moreover, it can be seen from Fig. 5 that \( \|x_0^* - x_\delta^*\|_2 < \|x_0^* - x_\delta^*\|_2 \) since \( x^* \) is the projection of \( x_\delta^* \) onto the unit sphere.

![Fig. 5. Visualization of \( \mathcal{F} \) and \( \mathcal{F}_2 \) and the relationship between \( x^*, x_0^*, \) and \( x_\delta^* \).](image)

Consequently, for a sufficiently small \( \delta \), the following holds
\[
\|x_\delta^* - x^*\|_2 < \delta^* \Rightarrow \|f(x_0^*) - f(x^*)\| < \epsilon^*,
\]
implying that \( x^* \) converges to the solution of (9). Since \( C \) is independent of \( x \), the original optimization problem in (9) may be approximated with
\[
\min_{x \in \mathcal{F}_2} \frac{f_1(x)}{f_2(x)}.
\] (13)

\[ \text{C. Quadratically Constrained Fractional Quadratic Program Solution} \]

The quadratically constrained fractional quadratic program was studied in [17]. Note that \( A_1 (A_1^*) \) and \( A_2 (A_2^*) \) are symmetric and there are no conditions on their definiteness. In case \( A_n (A_n^*) \) is not symmetric, it can be replaced by \( \frac{A_n + A_n^*}{2} (\frac{A_n^T + A_n}{2}) \) without changing the problem. In (13), \( A_2 > 0 (A_2^*) > 0 \), since it is a diagonal block of a positive definite matrix. The only assumption needed to solve (13) is that \( f_2(x) \) is bounded below on \( \mathcal{F}_2 \) by a positive number \( \gamma \). It can be seen from (5) and (7) that this assumption is trivially satisfied with \( f_2(x) \geq 1 \). An iterative bisection algorithm for obtaining an \( \epsilon \)-global optimal solution \( x^* \) for the problem (13) was developed in [17], specifically
\[
\alpha^* \leq \min_{x \in \mathcal{F}_2} \frac{f_1(x^*)}{f_2(x^*)} \leq \alpha^* + \epsilon,
\]
where \( x^* \in \mathcal{F}_2 \), \( \alpha^* \triangleq \min_{x \in \mathcal{F}_2} \{ f_1(x) \} \), and \( \epsilon \) is an arbitrarily small positive number. An upper and lower bound \( m \) and \( M \), respectively, on \( \min_{x \in \mathcal{F}_2} \{ f_1(x) \} \) must first be established. The following bounds may be established on \( f_1 \) and \( f_2 \)
\[
f_1(x) = \begin{cases} u^T Q u \leq 0, & f_2(x) = 1 + u^T M^{-1} u \geq 1, \end{cases}
\]
since \( Q \leq 0 \), \( Q^T \leq 0 \), and \( M^{-1} \succ 0 \). Subsequently, \( m \) and \( M \) may be chosen to be
\[
m \leq -\max_{x \in \mathcal{F}_2} f_1(x) = \begin{cases} -\max_{x \in \mathcal{F}_2} g_1(x), & -\max_{x \in \mathcal{F}_2} g_1'(x), \end{cases}
\]
\[
M = 0.
\] (14)
Noting that \( \|h_N\|^2_2 \leq 2 + \delta \), the following inequality holds
\[
0 \leq \|u\|_2 \leq M_N \sqrt{2 + \delta} + \frac{1}{\mu_N} \|H_N^{-1} y_N\|_2.
\]
Therefore, \( m \) may be chosen to be
\[
m = \min_{x \in \mathcal{F}_2} \left\{ \left[ \min_{x \in \mathcal{F}_2} \left[ \frac{1}{\mu_N} \|H_N^{-1} y_N\|_2 \right] \right]^2 \lambda_{\max} (-Q^T) \right\}.
\]
(15)
where \( \lambda_{\max} (-Q^T) \) denotes the largest eigenvalue. The following equivalence was shown in [22]
\[
\min_{x \in \mathcal{F}_2} \left\{ f_1(x) \right\} \leq \alpha \Rightarrow \min_{x \in \mathcal{F}_2} \{ f_1(x) - \alpha f_2(x) \} \leq 0.
\]
This equivalence enables (13) to be solved using a bisection algorithm, which is summarized in Algorithm 1.

**Algorithm 1 DOP Minimization**

1. **Given:** \( m, M \) (cf. (14) and (15)), and \( \epsilon \),
2. **Initialization:** \( l_0 = m, \quad u_0 = M \),
3. **Update:** \( \Delta_{ul} = 1 + \epsilon, \)
4. **while** \( \Delta_{ul} > \epsilon, \) \( k \geq 1 \), **do**
5. \( \alpha_k = \frac{l_k - u_k}{2}, \)
6. **Solve** minimize \( \beta_k = f_1(x) - \alpha_k f_2(x), \)
7. **if** \( \beta_k \leq 0 \) **then**
8. \( l_k \leftarrow l_{k-1}, \quad u_k \leftarrow \alpha_k, \)
9. **else**
10. \( l_k \leftarrow \alpha_k, \quad u_k \leftarrow u_{k-1}, \)
11. \( \Delta_{ul} \leftarrow u_k - l_k, \)
12. **Return** \( x^* = \arg\min_{x \in \mathcal{F}_2} \{ f_1(x) - u_k f_2(x) \}. \)

Next, the algorithm for minimizing \( f_1(x) - \alpha f_2(x) \) is described. It can be seen that minimizing \( f_1(x) - \alpha f_2(x) \) is equivalent to minimizing \( x^T \tilde{A} x - 2 \tilde{b}^T x + \tilde{c} \), where
\[
\tilde{A} \triangleq A_1 - \alpha A_2, \quad \tilde{b} \triangleq b_1 - \alpha b_2, \quad \tilde{c} \triangleq c_1 - \alpha c_2.
\]
In the case of minimizing \( g'(\mathbf{P}_N) \), \( A_n, b_n, \) and \( c_n \) are replaced by \( A'_n, b'_n, \) and \( c'_n \), respectively. Note that \( \tilde{A} \) is symmetric; therefore, it is diagonalizable with the following eigenvalue decomposition

\[
\tilde{A} = \mathbf{U} \Lambda \mathbf{U}^T,
\]

where \( \mathbf{U} \) is orthonormal and \( \Lambda \) is a diagonal matrix whose diagonal elements are the eigenvalues of \( \tilde{A} \), denoted \( \lambda_i \). The eigenvalues and eigenvectors of \( \tilde{A} \) are re-ordered such that \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \). With the change of variable \( \mathbf{x} \triangleq \mathbf{U} \mathbf{s} \) and defining \( \mathbf{w} \triangleq \mathbf{U}^T \mathbf{b} = [w_1, w_2, w_3]^T \), the following optimization problems are equivalent

\[
\begin{align*}
\text{minimize} & \quad f_1(\mathbf{x}) - \alpha f_2(\mathbf{x}) \\
\Leftrightarrow \quad & \text{minimize} \quad s^T \Lambda s - 2 \mathbf{w}^T s + \bar{c},
\end{align*}
\]

(16)

The solution of (16) is given by \( \mathbf{s}^* = [s^*_1, s^*_2, s^*_3]^T \) [17], with

\[
s^*_i = \frac{w_i}{\lambda_i - \eta^* - \xi^*},
\]

where

\[
(\eta^*, \xi^*) = \begin{cases} 
(\bar{\eta}, 0), & \text{if } \lambda_3 \leq 0 \\
(\bar{\eta}, 0), & \text{if } \lambda_3 > 0 \text{ and } h(\bar{\eta}, 0) > h(0, \xi) \\
(0, \xi), & \text{otherwise},
\end{cases}
\]

and \( \bar{\eta} \) and \( \bar{\xi} \) are the solutions to the optimization problems

\[
\begin{align*}
\text{maximize} & \quad h(\eta, 0) \\
\text{maximize} & \quad h(0, \xi),
\end{align*}
\]

(17) \hspace{1cm} (18)

respectively, where \( \lambda^*_3 \) is the left-hand limit of \( \lambda_3 \) and

\[
h(\eta, \xi) \triangleq - \sum_{i=1}^{3} \frac{w_i^2}{\lambda_i - \eta - \xi} + (1 + \delta)\eta + \xi + \bar{c}.
\]

The functions \( h_1(\eta) \triangleq h(\eta, 0) \) and \( h_2(\xi) \triangleq h(0, \xi) \) are called secular functions [22]. These functions are strictly concave for \( \eta, \xi < \lambda_3 \), making (17) and (18) convex optimization problems. Therefore, one may solve for \( h'_1(\eta) \equiv 0 \) (\( h'_2(\xi) \equiv 0 \)) using iterative methods (e.g., Newton’s method) and if \( \bar{\eta} \geq 0 \) (\( \bar{\xi} \leq 0 \)), set \( \bar{\eta} \equiv 0 \) (\( \bar{\xi} \equiv 0 \)).

Finally, \( x^*_3 \) is obtained from \( x^*_3 = \mathbf{U} \mathbf{s}^* \) and \( x^* \) is obtained from (12).

V. SIMULATION RESULTS

In this section, simulation results validating the proposed approach are presented. To this end, two sets of Monte Carlo (MC) simulations are performed. In the first set, the solution obtained with the proposed algorithm is plotted against the global optimal solution obtained by exhaustively sweeping the entire feasible space. In the second set, the solution obtained with the proposed algorithm is plotted against the solution obtained with a general purpose solver.

A. Proposed Algorithm versus Global Optimal Solution

In the first set of simulations, the two cost functions were evaluated for three cases: (1) the WGDOP, i.e., \( \mathbf{T} = \mathbf{T}' = [I_{4 \times 4}, 0_{2 \times 2}] \), (2) the WHDOP, i.e., \( \mathbf{T} = \mathbf{T}' = [I_{2 \times 2}, 0_{2 \times 2}] \), and (3) the WVDOP, i.e., \( \mathbf{T} = \mathbf{T}' = \mathbf{e}_3^T \). For each case, \( 10^4 \) MC runs were conducted for \( N = 6 \) and 8. The optimal solutions computed using the proposed approach, denoted \( g(x^*) \) and \( g'(\mathbf{P}_N) \), were plotted against the global optimal solutions, denoted \( g^*(\mathbf{P}_N) \) and \( g'^*(\mathbf{P}_N) \), obtained by exhaustively sweeping the entire feasible set, respectively (see Figs. 6 and 7). The positions of the \( N - 1 \) pre-deployed sensors were generated randomly by drawing \( N - 1 \) elevation angles from \( \mathcal{U} \left( -\frac{\pi}{4}, \frac{\pi}{4} \right) \) and \( N - 1 \) azimuth angles from \( \mathcal{U} (0, 2\pi) \), where \( \mathcal{U}(a, b) \) denotes the uniform distribution with support over \([a, b]\). The measurement noise covariance \( \mathbf{R}_N \) is also generated randomly at each iteration. It can be seen that the optimal solutions obtained by the proposed approach were identical to the optimal solution obtained by exhaustively sweeping the feasible set. Note that the WVDOP results for \( g'(\mathbf{P}_N) \) is not plotted since in the WVDOP problem, \( g(\mathbf{P}_N) = g'(\mathbf{P}_N) \).

B. Proposed Algorithm versus Nonlinear Numerical Optimization Solver Solution

In the second set of simulations, the two cost functions were evaluated for the same three cases. For each case, \( 10^4 \)
MC runs were conducted for \( N = 6 \) and 8. The optimal solutions computed using the proposed approach was plotted against the global optimal solutions denoted \( g_{\text{fmincon}}(P_N) \) and \( g'_{\text{fmincon}}(P_N) \) obtained by using MATLAB’s nonlinear numerical optimization solver fmincon (see Figs. 8 and 9). The MATLAB solver was initialized randomly. The position of the \( N - 1 \) pre-deployed sensors and the measurement noise covariance \( \mathbf{R}_N \) were generated the same way as in the first set of simulations. It can be seen that all MC simulation points lie either on or below the \( g(x^*) = g_{\text{fmincon}}(P_N) \) and \( g'(x^*) = g'_{\text{fmincon}}(P_N) \) lines, indicating that \( g(x^*) \leq g_{\text{fmincon}}(P_N) \) and \( g'(x^*) \leq g'_{\text{fmincon}}(P_N) \) for all MC runs. The proposed method outperforms MATLAB’s fmincon, since fmincon may converge to a local minimum instead of the global minimum.

C. Discussion

The simulation results presented in Subsections V-A and V-B reveal that while general-purpose nonlinear optimal solvers could converge to a local minimum, the proposed algorithm always converges to the global minimum, regardless of the configuration of the pre-deployed sensors (navigation sources) or the measurement noise covariance matrix.

Next, the complexity of the proposed algorithm is analyzed. First, the complexity of obtaining the global minimum by exhaustively sweeping the feasible space is discussed. Consider a uniform gridding of the elevation and azimuth angles. Denote \( S \) the resulting number of discrete intervals in the elevation angle range. Since the azimuth angle range is twice as large as the elevation angle range, the resulting number of discrete intervals in the azimuth angle range will be \( 2S \). Therefore, there will be \( 2S^2 \) feasible points to evaluate. However, the complexity of the proposed algorithm is independent of the gridding resolution. In [17], it is noted that the computationally
expensive part of the proposed algorithm is computing the eigenvalues of $\tilde{\mathbf{A}}$, which has a complexity of $\mathcal{O}(n^3)$, where $n$ is the size of the matrix. However, in the optimization problems addressed in this paper, the size of $\mathbf{A}$ is always 3, which means that the cost of the proposed algorithm is $\mathcal{O}(9)$ per iteration. Therefore, as $S$ increases, the number of feasible solutions increases quadratically, whereas the complexity of the proposed algorithm remains constant. Note that $\epsilon$ was chosen to be $\epsilon = 10^{-7}$. In order to obtain this resolution in the exhaustive sweeping approach, $S$ becomes impractically large. For $S = 128$, the sweeping algorithm ran, on average, 200 times slower than the proposed algorithm.

VI. CONCLUSIONS

This paper proposed a method for obtaining a global minimum for the DOP minimization problem. Two equivalent problems were formulated, where it was assumed that an $N$th sensor (navigation source) was to be added to a set of $N-1$ pre-deployed sensors (navigation sources) estimating the 3-D position and clock bias of a source (UAV). The additional sensor was to be placed so to minimize variations of the DOP problem (WGDO, WHDOP, WVDOP, etc.). It was shown that the proposed cost functions are nonconvex. Subsequently, a method for obtaining the global minimum of the proposed cost functions was presented by formulating the DOP minimization problem as a quadratically constrained fractional quadratic problem. Simulation results were provided validating the global optimality of the solution obtained from the proposed algorithm. Moreover, it was shown that the proposed method is superior to nonlinear numerical optimization solvers, that often converge to local optima.

REFERENCES