Dynamic Compressive Sensing:

Sparse recovery algorithms for streaming signals and video

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Dynamic compressive sensing

Part 1

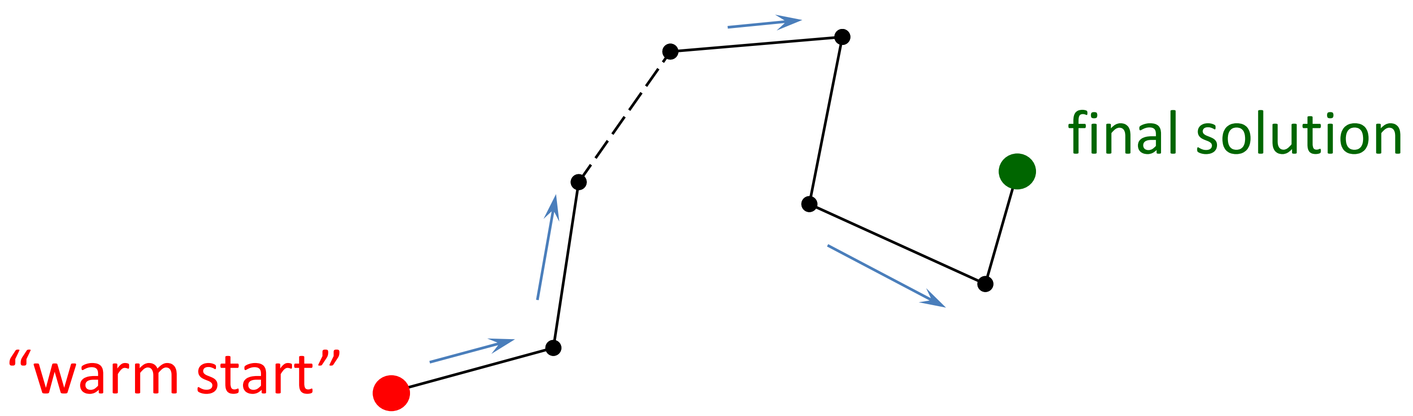
Dynamic updating

Use previous solutions and dynamics to quickly solve the recovery problems

Part 2

Dynamic modeling

Use the dynamical signal structure to improve the reconstruction



Dynamic compressive sensing

Part 1

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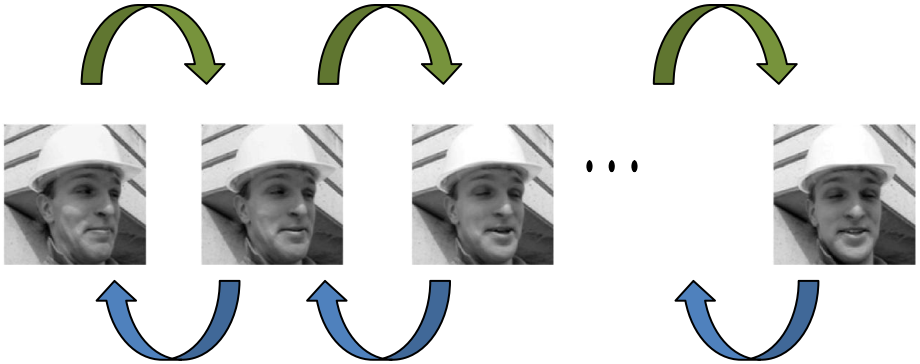
Use previous solutions and dynamics to accelerate the recovery process

Part 2

Dynamic modeling

Use the dynamical signal structure to improve the reconstruction



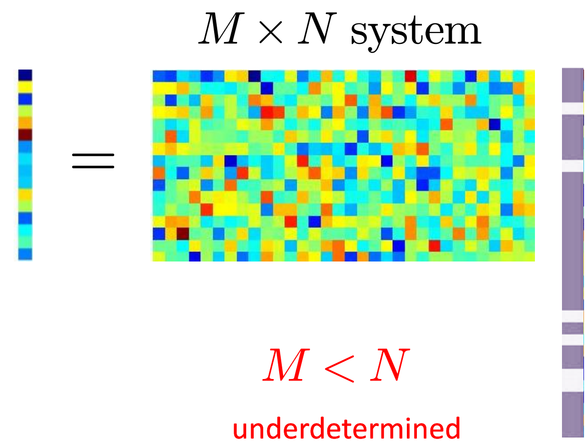
Compressive sensing

- A theory that deals with signal recovery from underdetermined systems:

$$y = \Phi x$$

Incoherent measurement matrix (e.g., subGaussian matrices) → Φ

Sparse signal → x



1. Compression with sampling
2. Sparse representation in a redundant dictionary

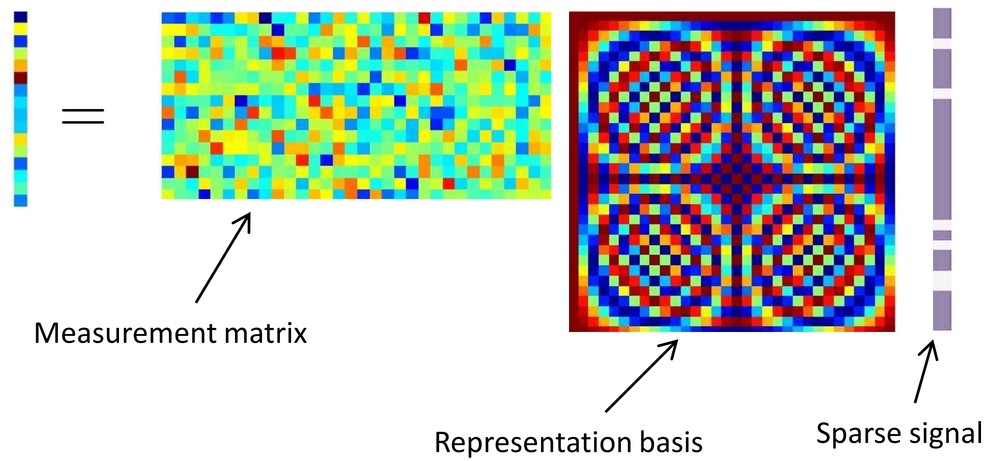
Similar principles: *structure and incoherence*

[Candes, Romberg, Tao, Donoho, Tropp, Baraniuk, Gilbert, Vershynin, Rudelson, …]

Compressive sensing

- Compressive sensing deals with signal recovery from underdetermined systems:

$$y = \Phi x \equiv \Phi \Psi \alpha$$



$$\text{minimize} \quad \tau \|\Psi^T x\|_1 + \frac{1}{2}\|\Phi x - y\|_2^2$$

sparsity

data fidelity

tradeoff parameter

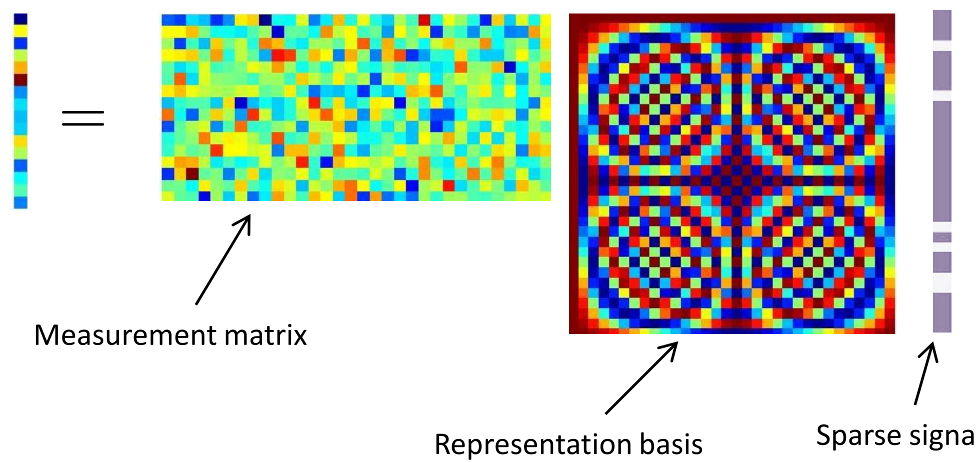
$$x = \Psi \alpha = \sum_{i=1}^{N} \psi_i \alpha_i$$

Sparse representation restricts the space of signals

Compressive sensing

- Compressive sensing deals with signal recovery from underdetermined systems:

$$y = \Phi x \equiv \Phi \Psi \alpha$$



$$\text{minimize} \; \| W \; \Psi^T x \|_1 + \frac{1}{2} \| \Phi x - y \|_2^2$$

sparsity *data fidelity*

Weights

$$x = \Psi \alpha = \sum_{i=1}^{N} \psi_i \alpha_i$$

Sparse representation restricts the space of signals

“Dynamic” compressive sensing

- “Static” compressive sensing:

$$y_t = \Phi_t x_t \equiv \Phi_t \Psi_t \alpha_t$$

dynamics

Fixed set of measurements — Fixed signal — Fixed model for representation

$$\text{minimize } \|W_t \Psi_t^T x_t\|_1 + \frac{1}{2}\|\Phi_t x_t - y_t\|_2^2$$

$x_1 \rightarrow x_2$ (innovations / dynamic model) $\Rightarrow$ $y_1 \rightarrow y_2$

$W_1 \rightarrow W_2$ Iterative reweighting

$\Psi_1 \rightarrow \Psi_2$ Data-adaptive model

$\Phi_1 \rightarrow \Phi_2$ Streaming measurements

“Dynamic” compressive sensing

- “Static” compressive sensing:

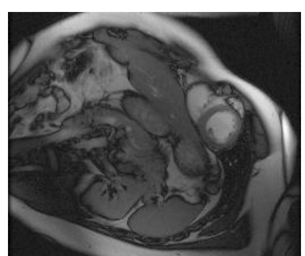
$$y_t = \Phi_t x_t \equiv \Phi_t \Psi_t \alpha_t$$

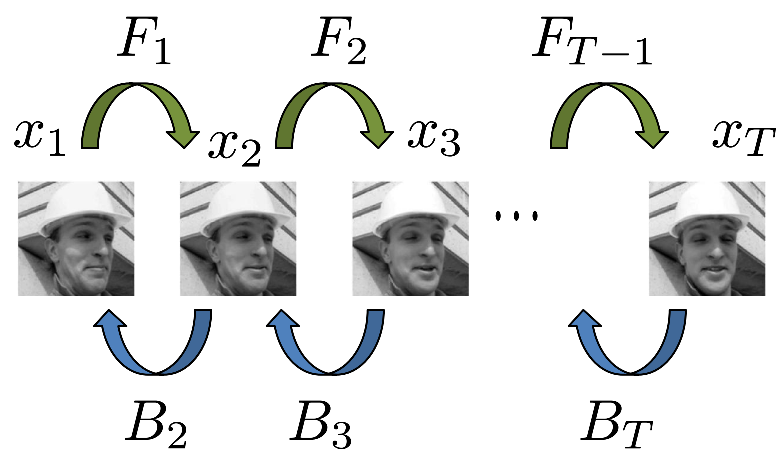
dynamics

Fixed set of measurements | Fixed signal | Fixed model for representation

$$\text{minimize } \|W_t \Psi_t^T x_t\|_1 + \frac{1}{2}\|\Phi_t x_t - y_t\|_2^2 + \|F_t x_t - x_{t+1}\|_p + \|B_t x_t - x_{t-1}\|_p$$

Spatio-temporal *structure (redundancies) in the videos as a dynamic model*



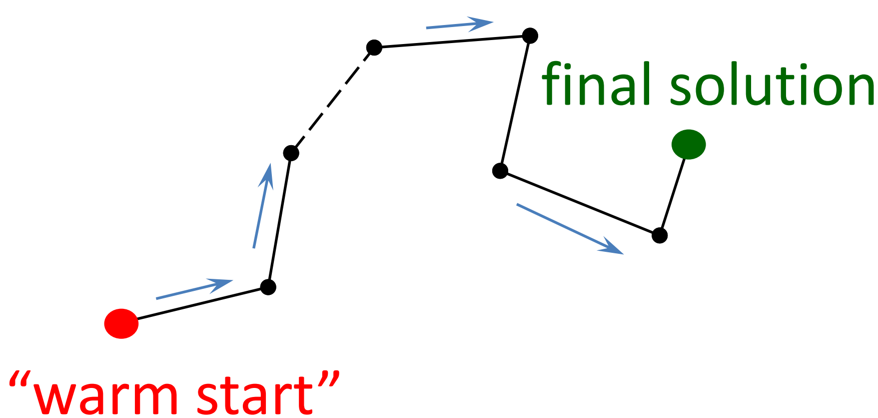


Dynamic compressive sensing

Dynamic updating

Quickly update the solution to accommodate changes

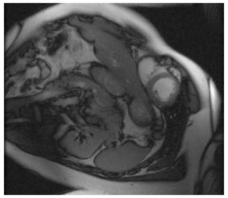
- ℓ_1 homotopy
- Variations:
  - Streaming signal
  - Streaming measurements
  - More…

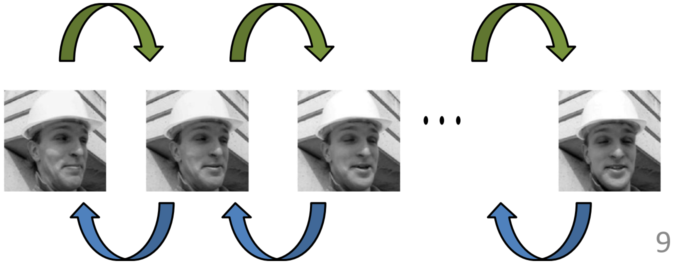


Dynamic modeling

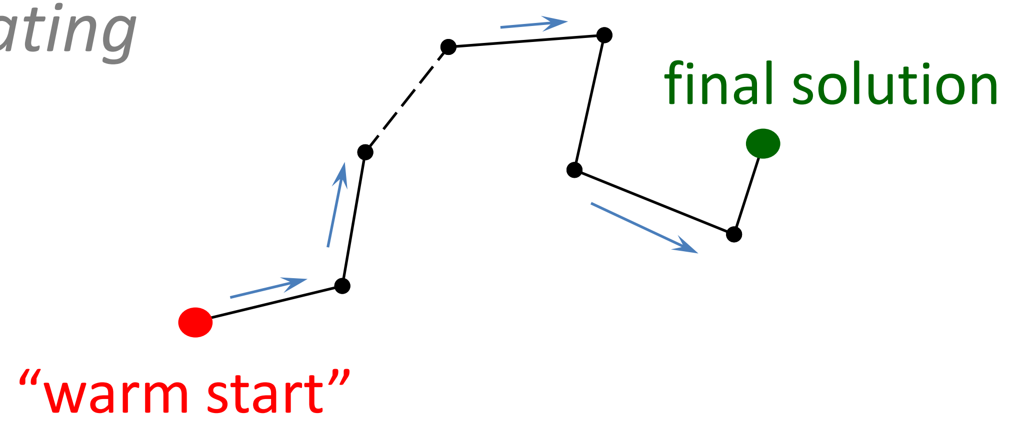
Improve reconstruction by exploiting the dynamical signal structure

1. Low-complexity video compression
2. Accelerated dynamic MRI





Part 1: Dynamic ℓ_1 updating

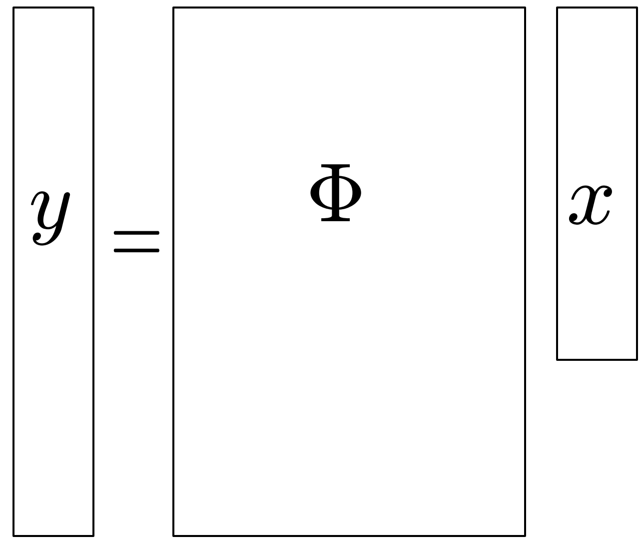


Motivation: dynamic updating in LS

- System model:

$$y = \Phi x$$

  - Φ is *full rank*
  - x is *arbitrary*



- LS estimate

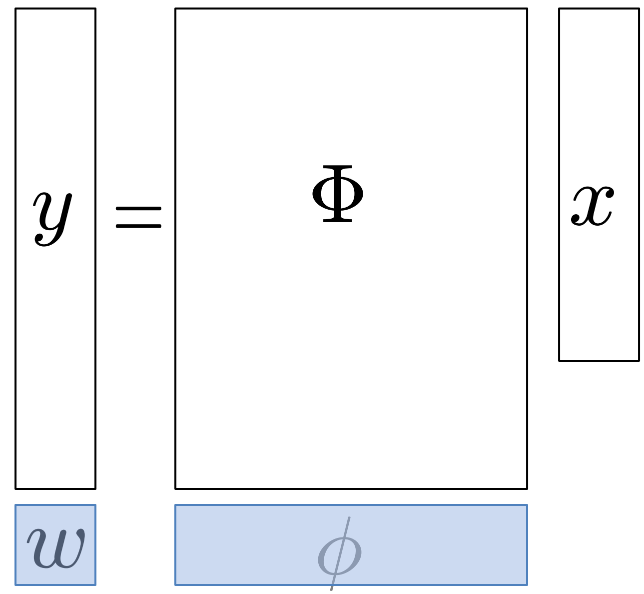
$$\text{minimize } \|\Phi x - y\|_2 \quad \rightarrow \quad x_0 = (\Phi^T \Phi)^{-1} \Phi^T y$$

- Updates for a time-varying signal with the same Φ mainly incurs a one-time cost of factorization.

Recursive updates

- Sequential measurements:

$$\begin{bmatrix} y \\ w \end{bmatrix} = \begin{bmatrix} \Phi \\ \phi \end{bmatrix} x$$



- Recursive LS

$$x_1 = (\Phi^T \Phi + \phi^T \phi)^{-1} (\Phi^T y + \phi^T w)$$

$$= x_0 + K_1 (w - \phi x_0)$$

Rank one update

$$K_1 = (\Phi^T \Phi)^{-1} \phi^T (1 + \phi (\Phi^T \Phi)^{-1} \phi^T)$$

Dynamics with the Kalman filter

- Linear dynamical system:

$$y_i = \Phi_i x_i + e_i$$
$$x_{i+1} = F_i x_i + f_i$$

$$\text{minimize} \sum_{i=1}^{k} \sigma_i \|\Phi_i x_i - y_i\|_2^2 + \lambda_i \|x_i - F_{i-1} x_{i-1}\|_2^2$$

Kalman filter equation

- *Update requires few low-rank updates*

Dynamic ℓ_1 updating

- Quickly update the solution of the ℓ_1 program as the system parameters change.

$$y = \Phi \bar{x} + e \quad \rightarrow \quad \text{minimize } \|Wx\|_1 + \frac{1}{2}\|\Phi x - y\|_2^2$$

- Variations:
  - Time-varying signal
  - Streaming measurements
  - Iterative reweighting
  - Streaming signals with overlapping measurements
  - Sparse signal in a linear dynamical system (*sparse Kalman filter*)

Dynamic ℓ_1 updating

- Quickly update the solution of the ℓ_1 program as the system parameters change.

$$y_t = \Phi_t \bar{x}_t + e_t \quad \rightarrow \quad \text{minimize } \|W_t x_t\|_1 + \frac{1}{2}\|\Phi_t x_t - y_t\|_2^2$$

variations

- Variations:
  - Time-varying signal
  - Streaming measurements
  - Iterative reweighting
  - Streaming signals with overlapping measurements
  - Sparse signal in a linear dynamical system (*sparse Kalman filter*)

Time-varying signals

- System model: $y_1 = \Phi x_1 + e_1$
- ℓ_1 problem: $\text{minimize } \tau\|x\|_1 + \frac{1}{2}\|\Phi x - y_1\|_2^2$
- Signal varies: $x_1 \rightarrow x_2 \quad \Rightarrow \quad y_1 \rightarrow y_2$

  "Sparse innovations"
- New ℓ_1 problem: $\text{minimize } \tau\|x\|_1 + \frac{1}{2}\|\Phi x - y_2\|_2^2$
- Homotopy: $\text{minimize } \tau\|x\|_1 + \frac{1}{2}\|\Phi x - (1-\epsilon)y_1 - \epsilon y_2\|_2^2$

  Homotopy parameter: $0 \rightarrow 1$

[A., Romberg, "Dynamic L1 updating", c. 2009]

Time-varying signals

- System model: $y_1 = \Phi x_1 + e_1$
- ℓ_1 problem: $\text{minimize } \tau\|x\|_1 + \frac{1}{2}\|\Phi x - y_1\|_2^2$
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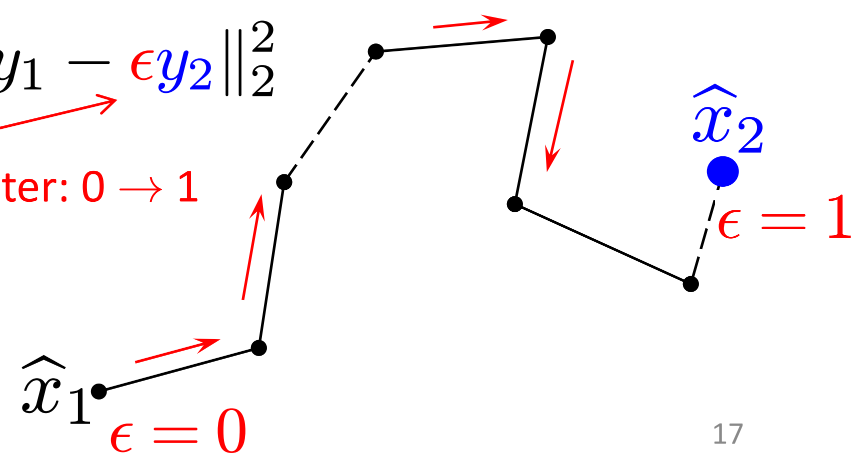
  "Sparse innovations"
- New ℓ_1 problem: $\text{minimize } \tau\|x\|_1 + \frac{1}{2}\|\Phi x - y_2\|_2^2$

$$\text{minimize } \tau\|x\|_1 + \frac{1}{2}\|\Phi x - (1-\epsilon)y_1 - \epsilon y_2\|_2^2$$

Homotopy parameter: 0 $\rightarrow$ 1



- *Path from old solution to new solution is* *piecewise linear* *and it is parameterized by* ϵ*: 0* $\rightarrow$ *1*

Time-varying signal

- Homotopy: $\text{minimize } \tau\|x\|_1 + \frac{1}{2}\|\Phi x - (1-\epsilon)y_1 - \epsilon y_2\|_2^2$

- Optimality conditions:
  Must be obeyed by any solution x^* with support Γ and sign sequence z

$$\Phi_\Gamma^T(\Phi x^* - (1-\epsilon)y_1 - \epsilon y_2) = -\tau z$$
$$\|\Phi_{\Gamma^c}^T(\Phi x^* - (1-\epsilon)y_1 - \epsilon y_2)\|_\infty < \tau$$

$$\begin{pmatrix} \tau g + \Phi^T(\Phi x^* - (1-\epsilon)y_1 - \epsilon y_2) = 0, \\ g = \partial\|x^*\|_1 : \quad \|g\|_\infty \leqslant 1,\ g^T x^* = \|x^*\|_1 \end{pmatrix}$$

Optimality: set subdifferential of the objective to zero

Time-varying signal

- Homotopy: $\text{minimize } \tau\|x\|_1 + \frac{1}{2}\|\Phi x - (1-\epsilon)y_1 - \epsilon y_2\|_2^2$
- Optimality conditions: $\Phi_\Gamma^T(\Phi x^* - (1-\epsilon)y_1 - \epsilon y_2) = -\tau z$
  Must be obeyed by any solution x^* with support Γ and sign sequence z: $\|\Phi_{\Gamma^c}^T(\Phi x^* - (1-\epsilon)y_1 - \epsilon y_2)\|_\infty < \tau$
- Change ϵ to $\epsilon+\delta$:

$$\Phi_\Gamma^T(\Phi x^* - (1-\epsilon)y_1 - \epsilon y_2) + \delta\Phi_\Gamma^T(\Phi\partial x + y_1 - y_2) = -\tau z$$

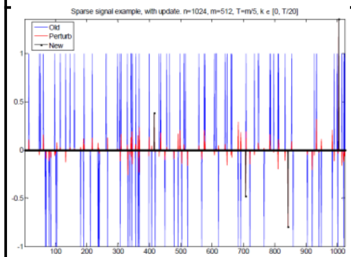
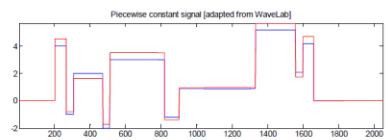
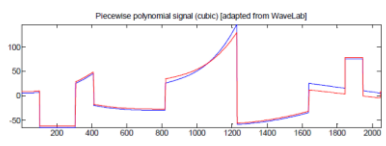
$$\|\Phi_{\Gamma^c}^T(\Phi x^* - (1-\epsilon)y_1 - \epsilon y_2) + \delta\Phi_{\Gamma^c}^T(\Phi\partial x + y_1 - y_2)\|_\infty < \tau$$

- **Update direction:**

$$\partial x = \begin{cases} (\Phi_\Gamma^T\Phi_\Gamma)^{-1}\Phi_\Gamma^T(y_2 - y_1), & \text{on } \Gamma \\ 0, & \text{on } \Gamma^c \end{cases}$$

Results

[A., Romberg, "Dynamic L1 updating", c. 2009]

Signal type	DynamicX (nProdAtA, CPU)	LASSO (nProdAtA, CPU)	GPSR-BB (nProdAtA, CPU)	FPC_AS (nProdAtA, CPU)
	(23.72, 0.132)	(235, 0.924)	(104.5, 0.18)	(148.65, 0.177)
	(2.7, 0.028)	(76.8, 0.490)	(17, 0.133)	(53.5, 0.196)
	(13.83, 0.151)	(150.2, 1.096)	(26.05, 0.212)	(66.89, 0.25)
	(26.2, 0.011)	(53.4, 0.019)	(92.24, 0.012)	(90.9, 0.036)

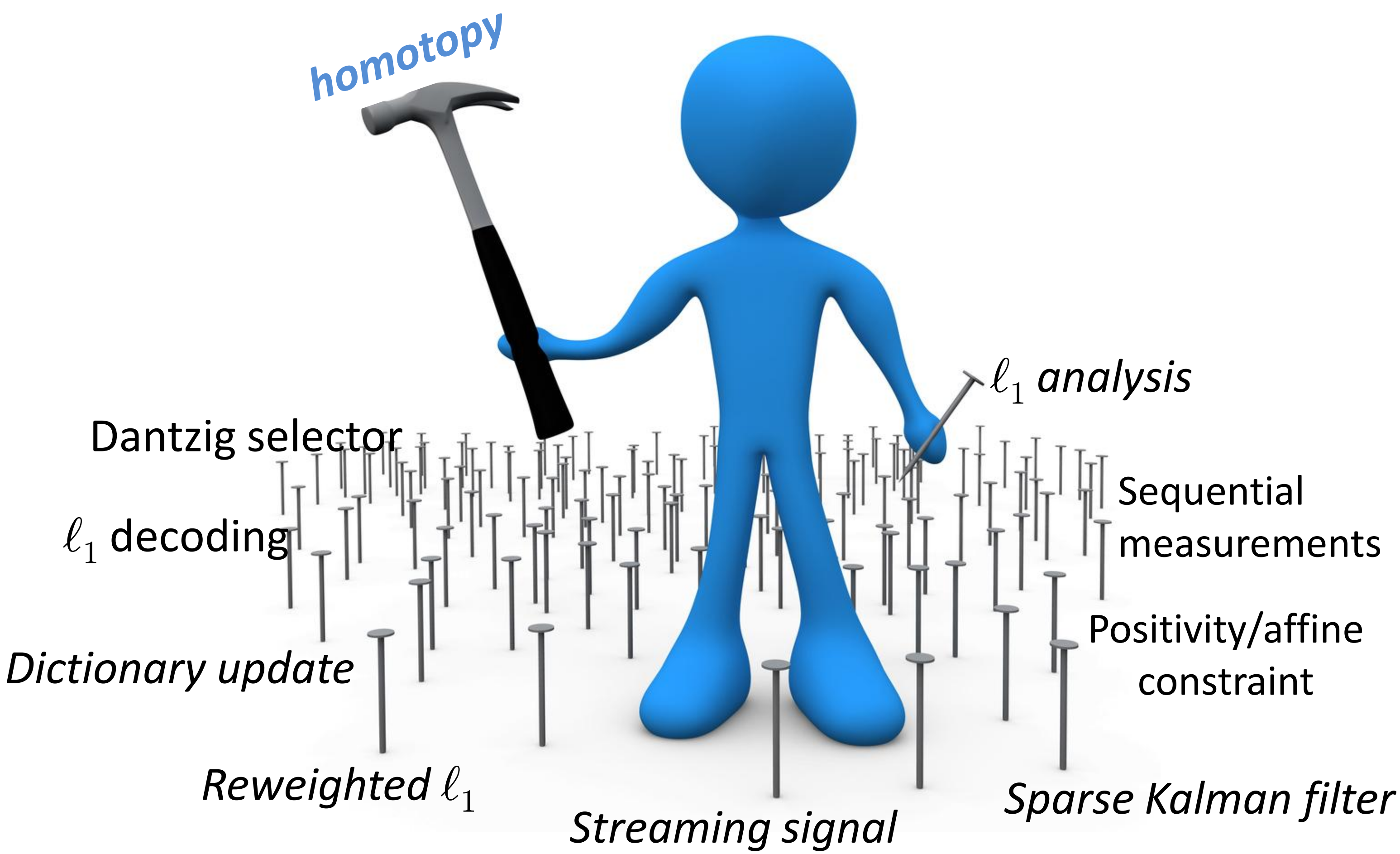
$\tau = 0.01\|A^T y\|_\infty$ nProdAtA: avg. number of matrix-vector products with Φ and Φ^T

CPU: average cputime to solve

GPSR: Gradient Projection for Sparse Reconstruction. M. Figueiredo, R. Nowak, S. Wright

FPC_AS: Fixed-point continuation and active set. Z. Wen, W. Yin, D. Goldfarb, and Y. Zhang

“If all you have is a hammer, everything looks like a nail.”



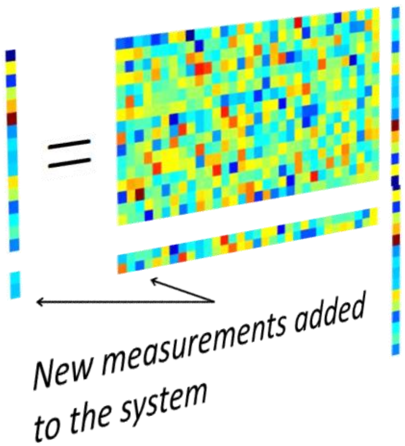
- Sequential measurements:

Homotopy parameter: $0 \rightarrow 1$

$$\text{minimize } \tau\|x\|_1 + \frac{1}{2}\|Ax - y\|_2^2 + \frac{1}{2}\|Bx - (1-\epsilon)\underline{B\widehat{x}_{old}} - \epsilon w\|_2^2$$

Dummy vector used to ensure optimality of previous solution with the changes

[A., Romberg, 2010]



ℓ_1 analysis

Dantzig selector

Sequential measurements

ℓ_1 decoding

Positivity/affine constraint

Dictionary update

Reweighted ℓ_1

Sparse Kalman filter

Streaming signal

Previous work on single measurement update: [Garrigues & El Ghaoui ’08; A., Romberg, ‘08]

- Iterative reweighting :

Homotopy parameter: $0 \rightarrow 1$

$$\text{minimize } \|[(1-\epsilon)W + \epsilon\,\widetilde{W}]x\|_1 + \frac{1}{2}\|\Phi x - y\|_2^2$$

Weights are updated

[A., Romberg, 2012]

Adaptive reweighting in Chapter VI in the thesis

ℓ_1 *analysis*

Dantzig selector

ℓ_1 decoding

Sequential measurements

Positivity/affine constraint

Dictionary update

Reweighted ℓ_1

Streaming signal

Sparse Kalman filter

"One *homotopy* to rule them all"

$$\text{minimize } \|\mathbf{W}\mathbf{x}\|_1 + \frac{1}{2}\|\boldsymbol{\Phi}\mathbf{x} - \mathbf{y}\|_2^2 + (1 - \epsilon)\mathbf{u}^T\mathbf{x}$$

[A., Romberg, 2013]

Dantzig selector

ℓ_1 decoding

Sequential measurements

Positivity/affine constraint

Dictionary update

Reweighted ℓ_1

Streaming signal

Sparse Kalman filter

ℓ_1-homotopy

- System model: $$\mathbf{y} = \mathbf{\Phi}\bar{\mathbf{x}} + \mathbf{e}$$

  $\bar{\mathbf{x}}$ is sparse

- Solve $$\underset{\mathbf{x}}{\text{minimize}} \; \|\mathbf{W}\mathbf{x}\|_1 + \frac{1}{2}\|\mathbf{\Phi}\mathbf{x} - \mathbf{y}\|_2^2$$

ℓ_1-homotopy

- System model: $\mathbf{y} = \mathbf{\Phi}\bar{\mathbf{x}} + \mathbf{e}$

  $\bar{\mathbf{x}}$ is sparse

- Instead, use the following versatile homotopy:

$$\underset{\mathbf{x}}{\text{minimize}} \; \|\mathbf{W}\mathbf{x}\|_1 + \frac{1}{2}\|\mathbf{\Phi}\mathbf{x} - \mathbf{y}\|_2^2 + (1-\epsilon)\,\mathbf{u}^{\mathbf{T}}\mathbf{x}$$

Optimization problem to solve

Homotopy part

$\hat{\mathbf{x}}$: *warm-start vector*

$\hat{\mathbf{z}} : \|\hat{\mathbf{z}}\|_\infty \leq 1, \hat{\mathbf{z}}^{\mathbf{T}}\hat{\mathbf{x}} = \|\hat{\mathbf{x}}\|_1$

$$\mathbf{u} \overset{\text{def}}{=} -\mathbf{W}\hat{\mathbf{z}} - \mathbf{\Phi}^{\mathbf{T}}(\mathbf{\Phi}\hat{\mathbf{x}} - \mathbf{y})$$

A “dummy” variable that maintains optimality of the starting point at $\epsilon = 0$

ℓ_1-homotopy

- System model: $\mathbf{y} = \mathbf{\Phi}\bar{\mathbf{x}} + \mathbf{e}$

$\bar{\mathrm{x}}$ is sparse

- A versatile homotopy formulation:

$$\underset{\mathbf{x}}{\text{minimize}} \; \|\mathbf{W}\mathbf{x}\|_1 + \frac{1}{2}\|\mathbf{\Phi}\mathbf{x} - \mathbf{y}\|_2^2 + (1-\epsilon)\,\mathbf{u}^{\mathbf{T}}\mathbf{x}$$

Optimization problem to solve

Homotopy part

$$\mathbf{u} \stackrel{\text{def}}{=} -\mathbf{W}\hat{\mathbf{z}} - \mathbf{\Phi}^{\mathbf{T}}(\mathbf{\Phi}\hat{\mathbf{x}} - \mathbf{y})$$

$$\begin{pmatrix} \mathbf{W}\mathbf{g} + \mathbf{\Phi}^{\mathbf{T}}(\mathbf{\Phi}\mathbf{x}^* - \mathbf{y}) + (\mathbf{1}-\epsilon)\mathbf{u} = 0, \\ \mathbf{g} = \partial\|\mathbf{x}^*\|_\mathbf{1} : \quad \|\mathbf{g}\|_\infty \leqslant \mathbf{1},\; \mathbf{g}^{\mathbf{T}}\mathbf{x}^* = \|\mathbf{x}^*\|_\mathbf{1} \end{pmatrix}$$

Optimality: set subdifferential of the objective to zero

A "dummy" variable that maintains optimality of the starting point at $\epsilon = 0$

ℓ_1-homotopy

- Homotopy: $\text{minimize } \|\mathbf{Wx}\|_1 + \frac{1}{2}\|\mathbf{\Phi x} - \mathbf{y}\|_2^2 + (\mathbf{1} - \epsilon)\mathbf{u}^\mathbf{T}\mathbf{x}$

- Optimality conditions:

  Must be obeyed by any solution x*
  with support Γ and sign sequence z

$$\mathbf{\Phi}_\mathbf{\Gamma}^\mathbf{T}(\mathbf{\Phi x}^* - \mathbf{y}) + (\mathbf{1} - \epsilon)\mathbf{u} = -\mathbf{Wz}$$

$$|\mathbf{\Phi}^\mathbf{T}(\mathbf{\Phi x}^* - \mathbf{y}) + (\mathbf{1} - \epsilon)\mathbf{u}| \leqslant \mathbf{w}$$

ℓ_1-homotopy

- Homotopy: $\text{minimize } \|\mathbf{W}\mathbf{x}\|_1 + \frac{1}{2}\|\mathbf{\Phi}\mathbf{x} - \mathbf{y}\|_2^2 + (1-\epsilon)\mathbf{u}^T\mathbf{x}$

- Optimality conditions:

  Must be obeyed by any solution x^* with support Γ and sign sequence z

$$\mathbf{\Phi}_{\Gamma}^{T}(\mathbf{\Phi}\mathbf{x}^* - \mathbf{y}) + (1-\epsilon)\mathbf{u} = -\mathbf{W}\mathbf{z}$$

$$|\mathbf{\Phi}^{T}(\mathbf{\Phi}\mathbf{x}^* - \mathbf{y}) + (1-\epsilon)\mathbf{u}| \leqslant \mathbf{w}$$

- Change ϵ to $\epsilon+\delta$:

$$\mathbf{\Phi}_{\Gamma}^{T}(\mathbf{\Phi}\mathbf{x}^* - \mathbf{y}) + (1-\epsilon)\mathbf{u} + \delta\left(\mathbf{\Phi}_{\Gamma}^{T}\mathbf{\Phi}\partial\mathbf{x} - \mathbf{u}\right) = -\mathbf{W}\mathbf{z}$$

$$\Big|\underbrace{\phi_{\mathbf{i}}^{T}(\mathbf{\Phi}\mathbf{x}^* - \mathbf{y}) + (1-\epsilon)\mathbf{u}_{\mathbf{i}}}_{\mathbf{p}_i} + \delta\underbrace{\left(\phi_{\mathbf{i}}^{T}\mathbf{\Phi}\partial\mathbf{x} - \mathbf{u}_{\mathbf{i}}\right)}_{\mathbf{d}_i}\Big| \leqslant \mathbf{w}_{\mathbf{i}}$$

ℓ_1-homotopy

- Homotopy: $\text{minimize } \|\mathbf{W}\mathbf{x}\|_1 + \frac{1}{2}\|\mathbf{\Phi}\mathbf{x} - \mathbf{y}\|_2^2 + (1-\epsilon)\mathbf{u}^T\mathbf{x}$

- Optimality conditions:

  Must be obeyed by any solution x^* with support Γ and sign sequence z

$$\mathbf{\Phi}_\Gamma^T(\mathbf{\Phi}\mathbf{x}^* - \mathbf{y}) + (1-\epsilon)\mathbf{u} = -\mathbf{W}\mathbf{z}$$

$$|\mathbf{\Phi}^T(\mathbf{\Phi}\mathbf{x}^* - \mathbf{y}) + (1-\epsilon)\mathbf{u}| \leqslant \mathbf{w}$$

- Change ϵ to $\epsilon + \delta$:

$$\mathbf{\Phi}_\Gamma^T(\mathbf{\Phi}\mathbf{x}^* - \mathbf{y}) + (1-\epsilon)\mathbf{u} + \delta\,(\mathbf{\Phi}_\Gamma^T\mathbf{\Phi}\partial\mathbf{x} - \mathbf{u}) = -\mathbf{W}\mathbf{z}$$

$$|\underbrace{\phi_i^T(\mathbf{\Phi}\mathbf{x}^* - \mathbf{y}) + (1-\epsilon)\mathbf{u}_i}_{\mathbf{p}_i} + \delta\,\underbrace{(\phi_i^T\mathbf{\Phi}\partial\mathbf{x} - \mathbf{u}_i)}_{\mathbf{d}_i}| \leqslant \mathbf{w}_i$$

- Update direction:

$$\partial\mathbf{x} = \begin{cases} (\mathbf{\Phi}_\Gamma^T\mathbf{\Phi}_\Gamma)^{-1}\mathbf{u}_\Gamma, & \text{on } \Gamma \\ 0, & \text{on } \Gamma^c \end{cases}$$

ℓ_1-homotopy

- Homotopy: $\text{minimize } \|\mathbf{Wx}\|_1 + \frac{1}{2}\|\mathbf{\Phi x} - \mathbf{y}\|_2^2 + (1-\epsilon)\mathbf{u}^T\mathbf{x}$

- Optimality conditions: $\mathbf{\Phi}_\Gamma^T(\mathbf{\Phi x}^* - \mathbf{y}) + (1-\epsilon)\mathbf{u} = -\mathbf{Wz}$

  Must be obeyed by any solution x^* with support Γ and sign sequence z

  $$|\mathbf{\Phi}^T(\mathbf{\Phi x}^* - \mathbf{y}) + (1-\epsilon)\mathbf{u}| \leqslant \mathbf{w}$$

- Change ϵ to $\epsilon + \delta$:

$$\mathbf{\Phi}_\Gamma^T(\mathbf{\Phi x}^* - \mathbf{y}) + (1-\epsilon)\mathbf{u} + \delta\,(\mathbf{\Phi}_\Gamma^T\mathbf{\Phi}\partial\mathbf{x} - \mathbf{u}) = -\mathbf{Wz}$$

$$|\underbrace{\phi_i^T(\mathbf{\Phi x}^* - \mathbf{y}) + (1-\epsilon)\mathbf{u}_i}_{\mathbf{p}_i} + \delta\,\underbrace{(\phi_i^T\mathbf{\Phi}\partial\mathbf{x} - \mathbf{u}_i)}_{\mathbf{d}_i}| \leqslant \mathbf{w}_i$$

- **Step size:**

$$\delta^* = \min(\delta^+, \delta^-)$$

$$\delta^+ = \min_{i \in \Gamma^c}\left(\frac{\mathbf{w}_i - \mathbf{p}_i}{\mathbf{d}_i}, \frac{-\mathbf{w}_i - \mathbf{p}_i}{\mathbf{d}_i}\right)_+$$

$$\delta^- = \min_{i \in \Gamma}\left(\frac{-\mathbf{x}_i^*}{\partial\mathbf{x}_i}\right)_+$$

ℓ_1-homotopy

- Homotopy: $\quad \text{minimize } \|\mathbf{W}\mathbf{x}\|_1 + \frac{1}{2}\|\mathbf{\Phi}\mathbf{x} - \mathbf{y}\|_2^2 + (1-\epsilon)\mathbf{u}^T\mathbf{x}$
- Change ϵ to $\epsilon + \delta$:

$$\mathbf{\Phi}_\Gamma^T(\mathbf{\Phi}\mathbf{x}^* - \mathbf{y}) + (1-\epsilon)\mathbf{u} + \delta\,(\mathbf{\Phi}_\Gamma^T\mathbf{\Phi}\partial\mathbf{x} - \mathbf{u}) = -\mathbf{W}\mathbf{z}$$

$$|\underbrace{\phi_i^T(\mathbf{\Phi}\mathbf{x}^* - \mathbf{y}) + (1-\epsilon)\mathbf{u}_i}_{\mathbf{p}_i} + \delta\,\underbrace{(\phi_i^T\mathbf{\Phi}\partial\mathbf{x} - \mathbf{u}_i)}_{\mathbf{d}_i}| \leqslant \mathbf{w}_i$$

$$\partial\mathbf{x} = \begin{cases} (\mathbf{\Phi}_\Gamma^T\mathbf{\Phi}_\Gamma)^{-1}\mathbf{u}_\Gamma, & \text{on } \Gamma \\ 0, & \text{on } \Gamma^c \end{cases}$$

γ^+ *enters* Γ $\quad \delta^+ = \min_{i\in\Gamma^c}\left(\frac{\mathbf{w}_i - \mathbf{p}_i}{\mathbf{d}_i}, \frac{-\mathbf{w}_i - \mathbf{p}_i}{\mathbf{d}_i}\right)_+$

γ^- *leaves* Γ $\quad \delta^- = \min_{i\in\Gamma}\left(\frac{-\mathbf{x}_i^*}{\partial\mathbf{x}_i}\right)_+$

$$\delta^* = \min(\delta^+, \delta^-)$$

$\mathbf{x} \succeq 0$

- **Update:** $\quad \mathbf{x}^* \leftarrow \mathbf{x}^* + \delta^*\partial\mathbf{x},\ \epsilon \leftarrow \epsilon + \delta^*,\ \Gamma.$
- **Repeat until** $\quad \epsilon = 1$

The Dantzig selector ℓ_1 -homotopy

- Dantzig selector:

Primal

$$\begin{aligned} &\text{minimize } \|\mathbf{Wx}\|_1 \\ &\text{subject to } |\mathbf{\Phi^T}(\mathbf{\Phi x} - \mathbf{y})| \preceq \mathbf{q} \end{aligned}$$

Dual

$$\begin{aligned} &\text{maximize } -\lambda^{\mathbf{T}}\mathbf{\Phi^T y} - \|\mathbf{Q}\lambda\|_1 \\ &\text{subject to } |\mathbf{\Phi^T \Phi}\lambda| \preceq \mathbf{w}, \end{aligned}$$

- Primal-dual ℓ_1-homotopy:

Primal homotopy

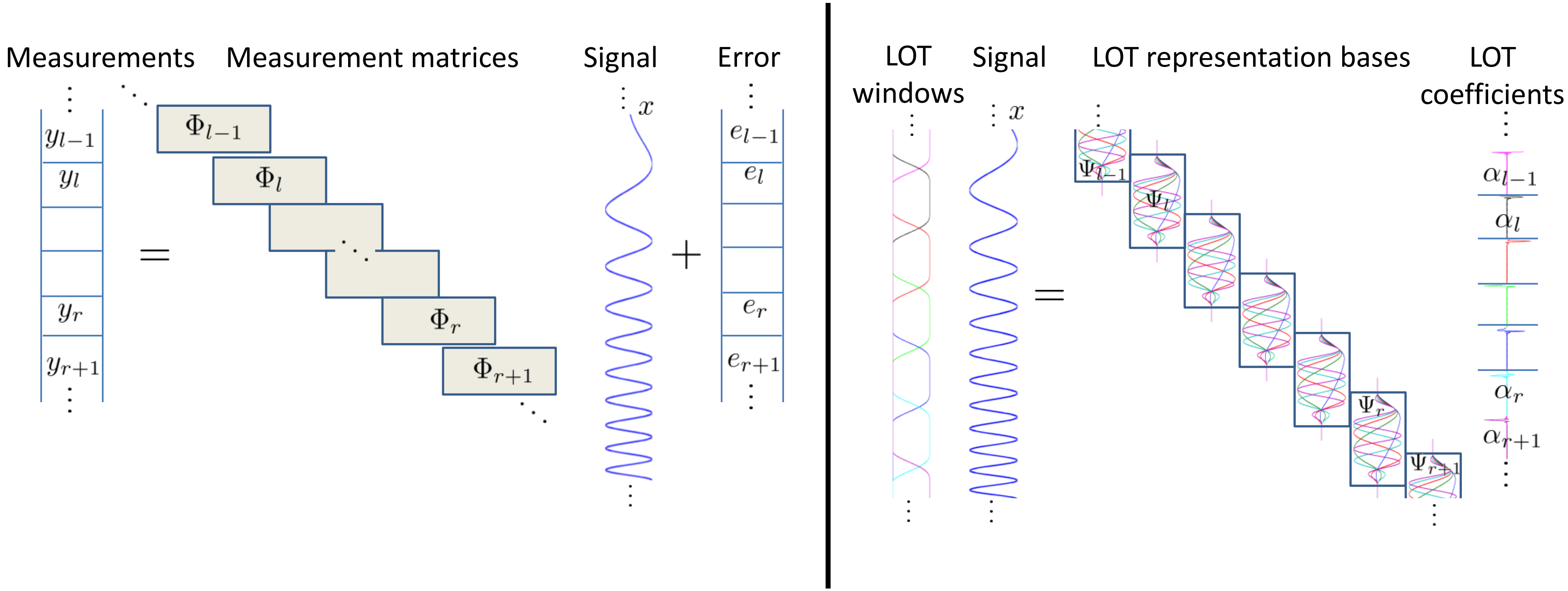
$$\begin{aligned} &\text{minimize } \|\mathbf{Wx}\|_1 + (\mathbf{1} - \epsilon)\mathbf{u^T x} \\ &\text{subject to } |\mathbf{\Phi^T}(\mathbf{\Phi x} - \mathbf{y}) + (\mathbf{1} - \epsilon)\mathbf{v}| \preceq \mathbf{q} \end{aligned}$$

Dual homotopy

$$\begin{aligned} &\text{maximize } -\lambda^{\mathbf{T}}(\mathbf{\Phi^T y} - (\mathbf{1} - \epsilon)\mathbf{v}) - \|\mathbf{Q}\lambda\|_1 \\ &\text{subject to } |\mathbf{\Phi^T \Phi}\lambda + (\mathbf{1} - \epsilon)\mathbf{u}| \preceq \mathbf{w}, \end{aligned}$$

Sparse recovery: streaming system

- Signal observations: $y_t = \Phi_t x_t + e_t$
- Sparse representation: $x[n] = \sum_{p,k} \alpha_{p,k} \psi_{p,k}[n]$

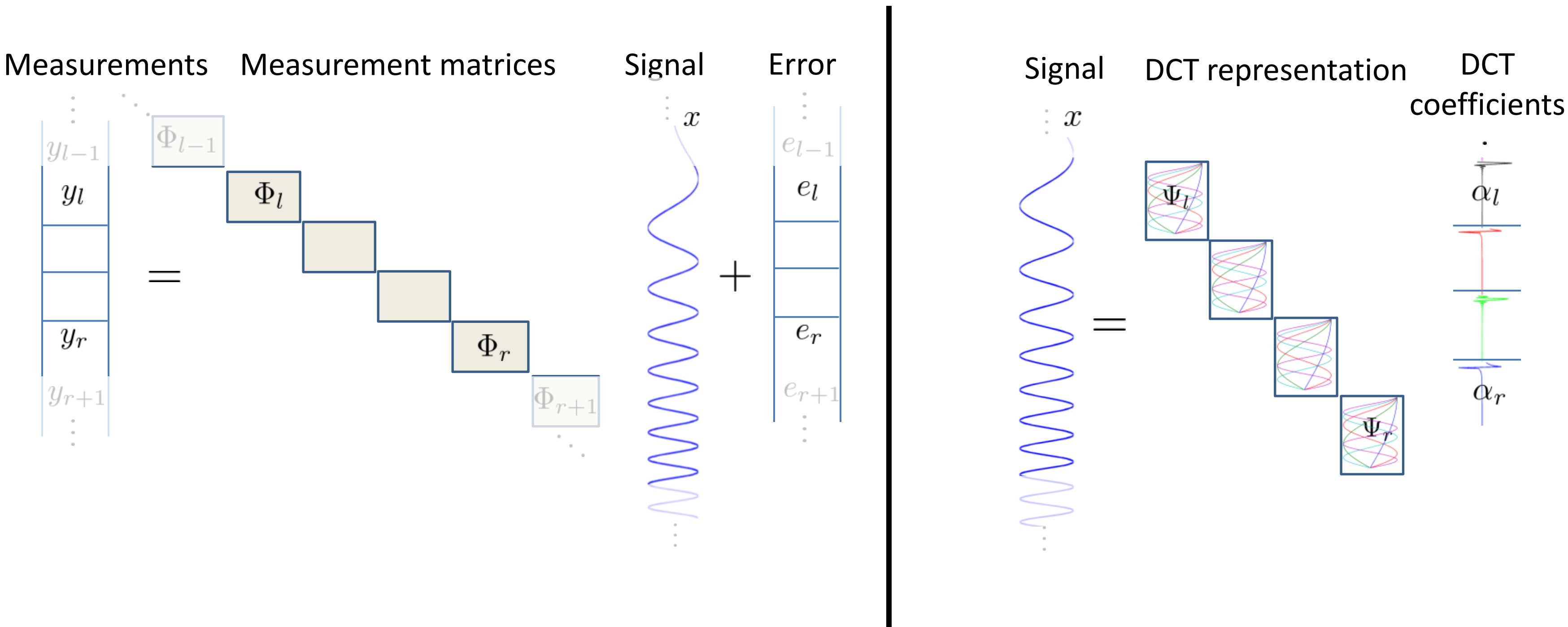


Sparse recovery: streaming system

- Time-varying signal *represented with block bases*

AND

- Streaming, *disjoint* measurements

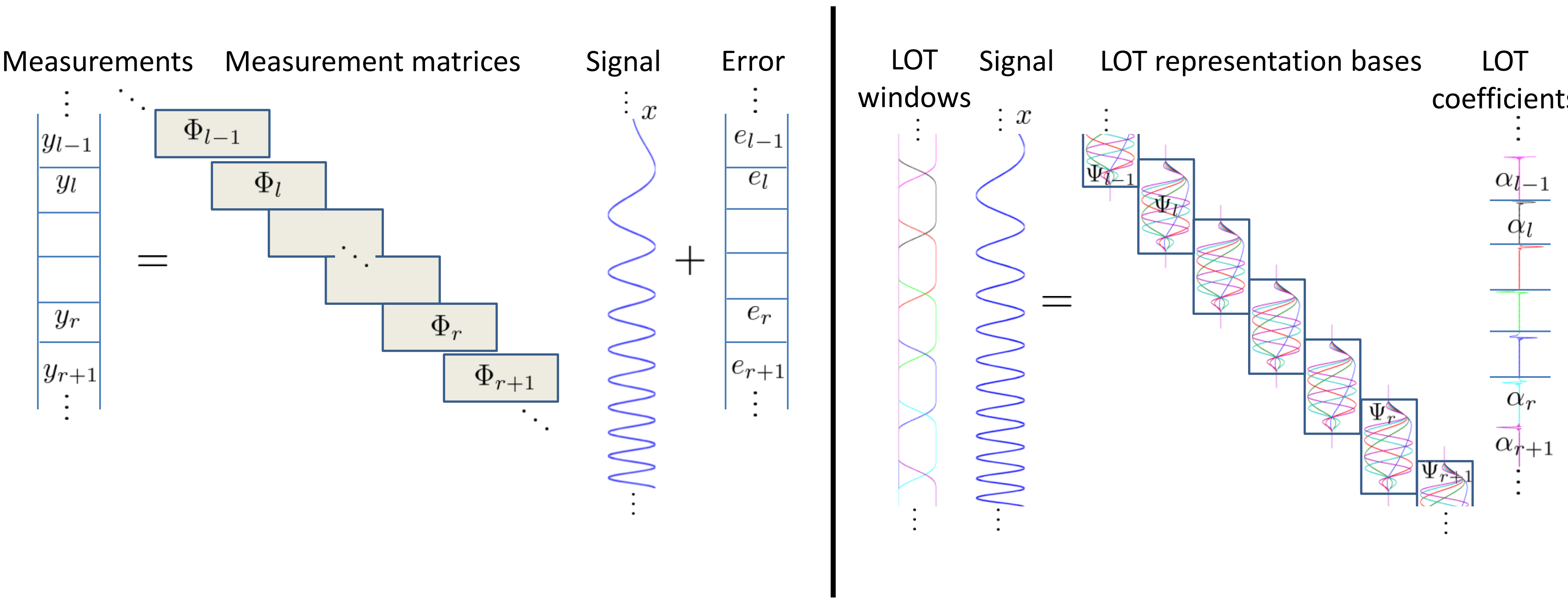


Sparse recovery: streaming system

- Time-varying signal *represented with lapped bases*

  OR

- Streaming, overlapping measurements

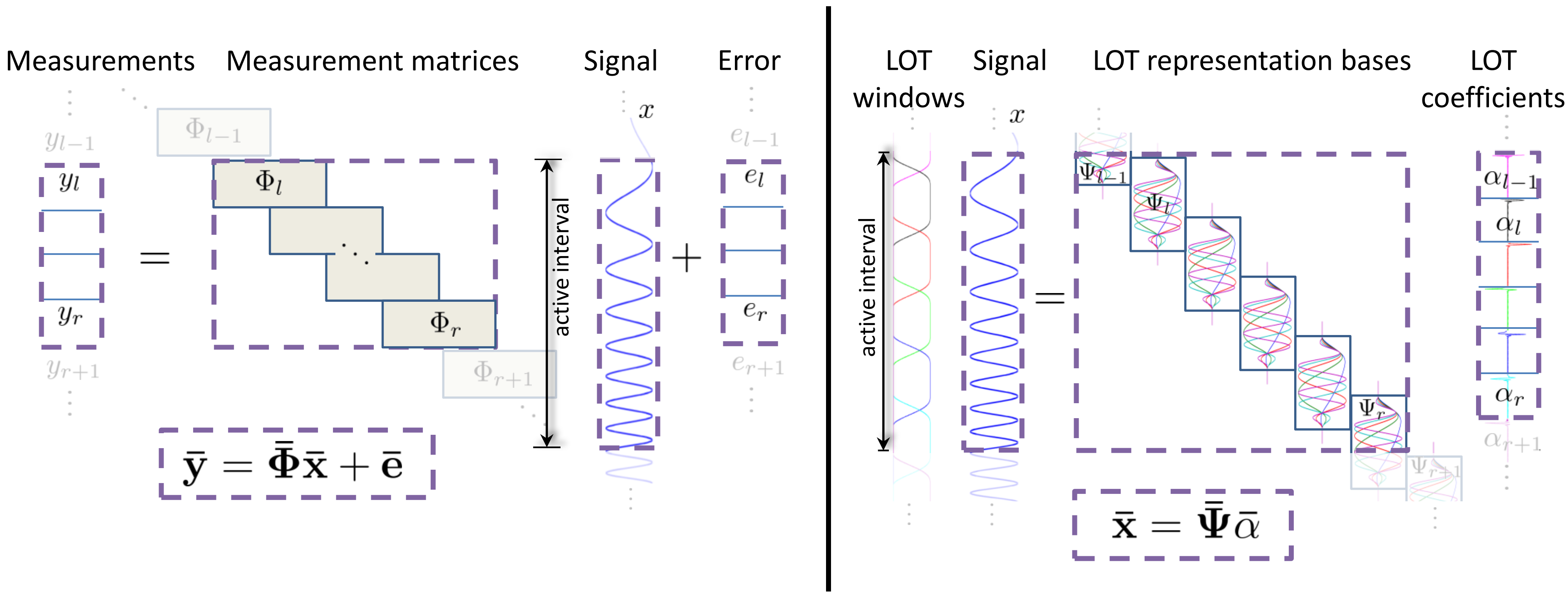


Sparse recovery: streaming system

- Time-varying signal *represented with lapped bases*

  OR

- Streaming, overlapping measurements



Sparse recovery: streaming system

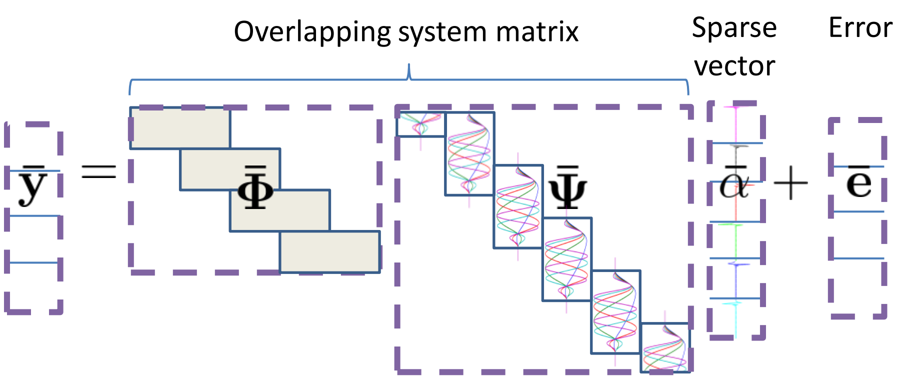
- Iteratively estimate the signal over a sliding (active) interval:

Desired

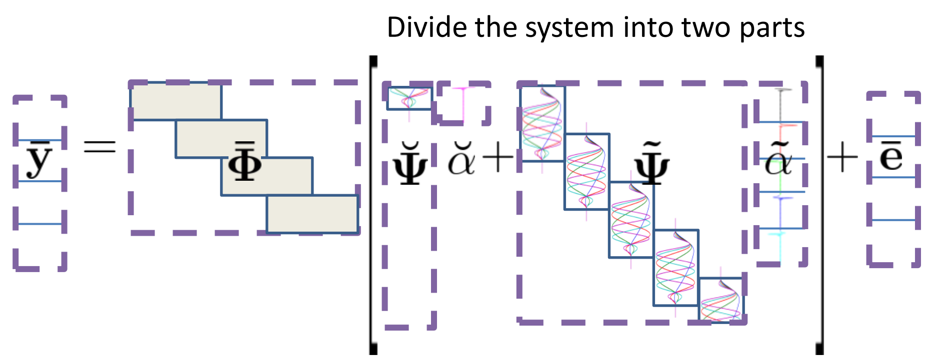
$$\text{minimize } \|\mathbf{W}\alpha\|_1 + \frac{1}{2}\|\bar{\mathbf{\Phi}}\tilde{\mathbf{\Psi}}\alpha - \tilde{\mathbf{y}}\|_2^2$$

Homotopy

$$\text{minimize } \|\mathbf{W}\alpha\|_1 + \frac{1}{2}\|\bar{\mathbf{\Phi}}\tilde{\mathbf{\Psi}}\alpha - \tilde{\mathbf{y}}\|_2^2 + (1-\epsilon)\mathbf{u}^T\alpha$$

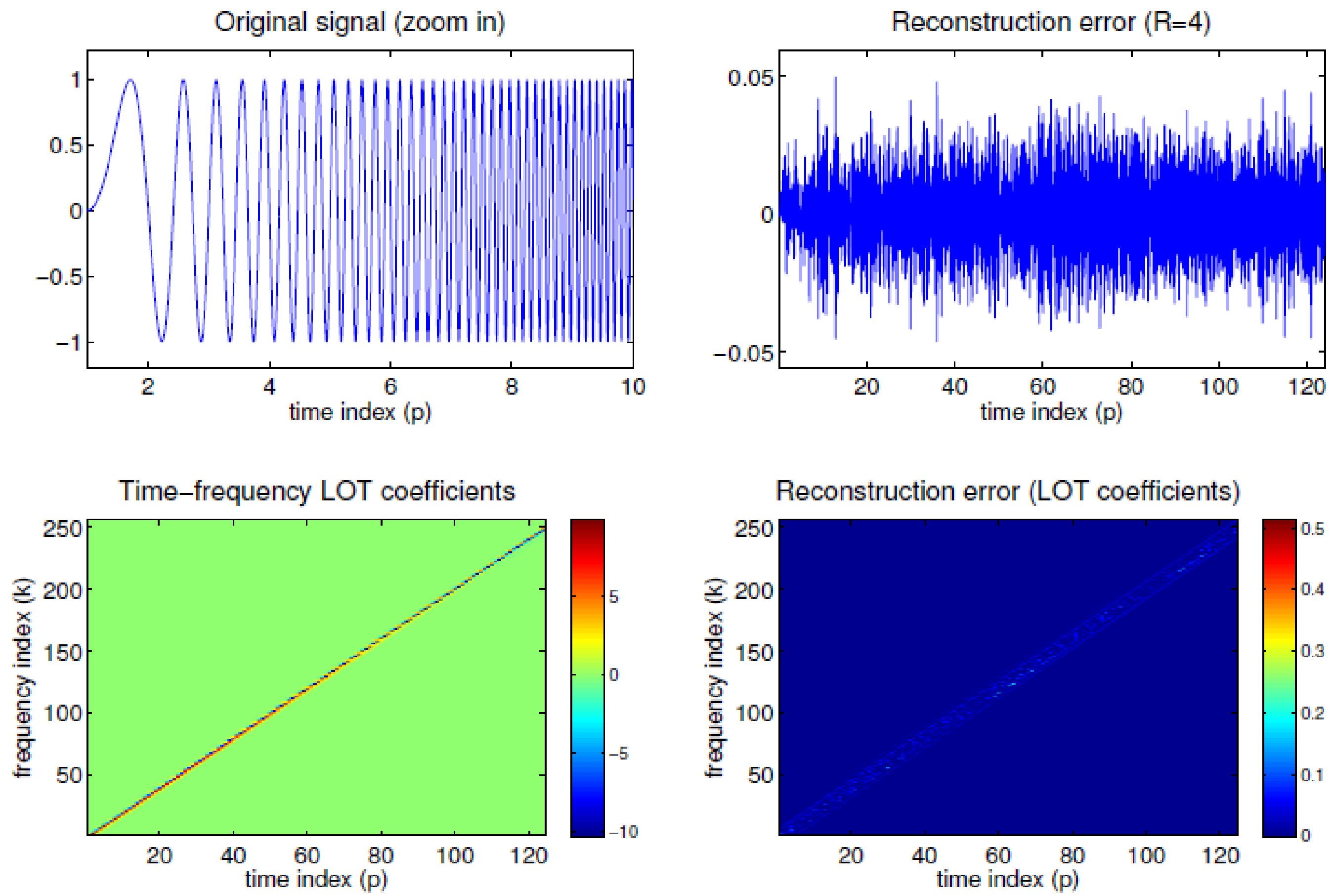


$$\bar{\mathbf{y}} = \bar{\mathbf{\Phi}}\bar{\mathbf{\Psi}}\bar{\alpha} + \bar{\mathbf{e}}$$



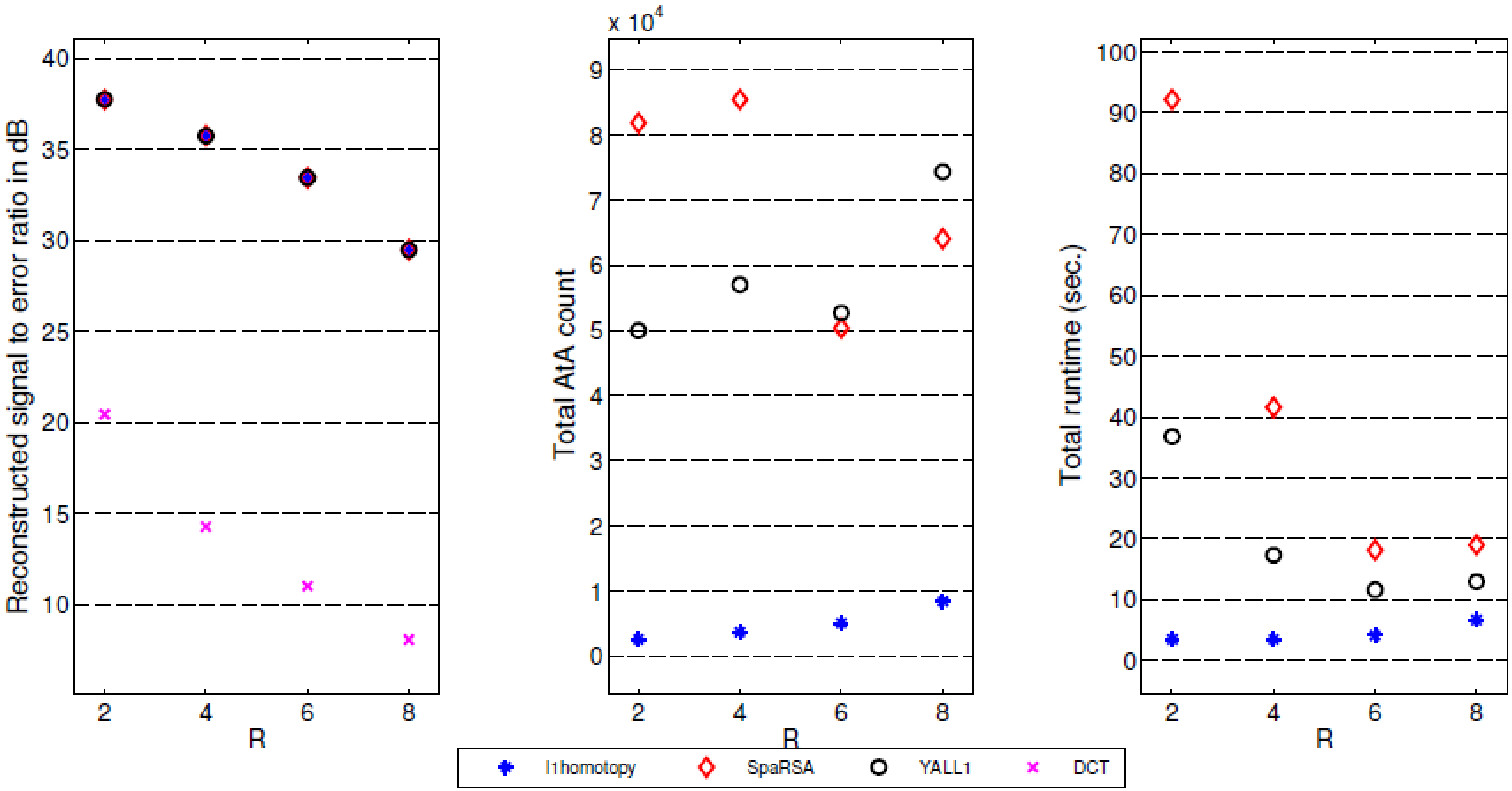
$$\bar{\mathbf{y}} = \bar{\mathbf{\Phi}}[\breve{\mathbf{\Psi}}\breve{\alpha} + \tilde{\mathbf{\Psi}}\tilde{\alpha}] + \bar{\mathbf{e}} \Rightarrow \tilde{\mathbf{y}} = \bar{\mathbf{y}} - \bar{\mathbf{\Phi}}\breve{\mathbf{\Psi}}\breve{\alpha}$$

Streaming signal recovery - Results



(Top-left) Linear chirp signal (zoomed in for first 2560 samples.
(Top-right) Error in the reconstruction at R=N/M = 4.
(Bottom-left) LOT coefficients. (Bottom-right) Error in LOT coefficients

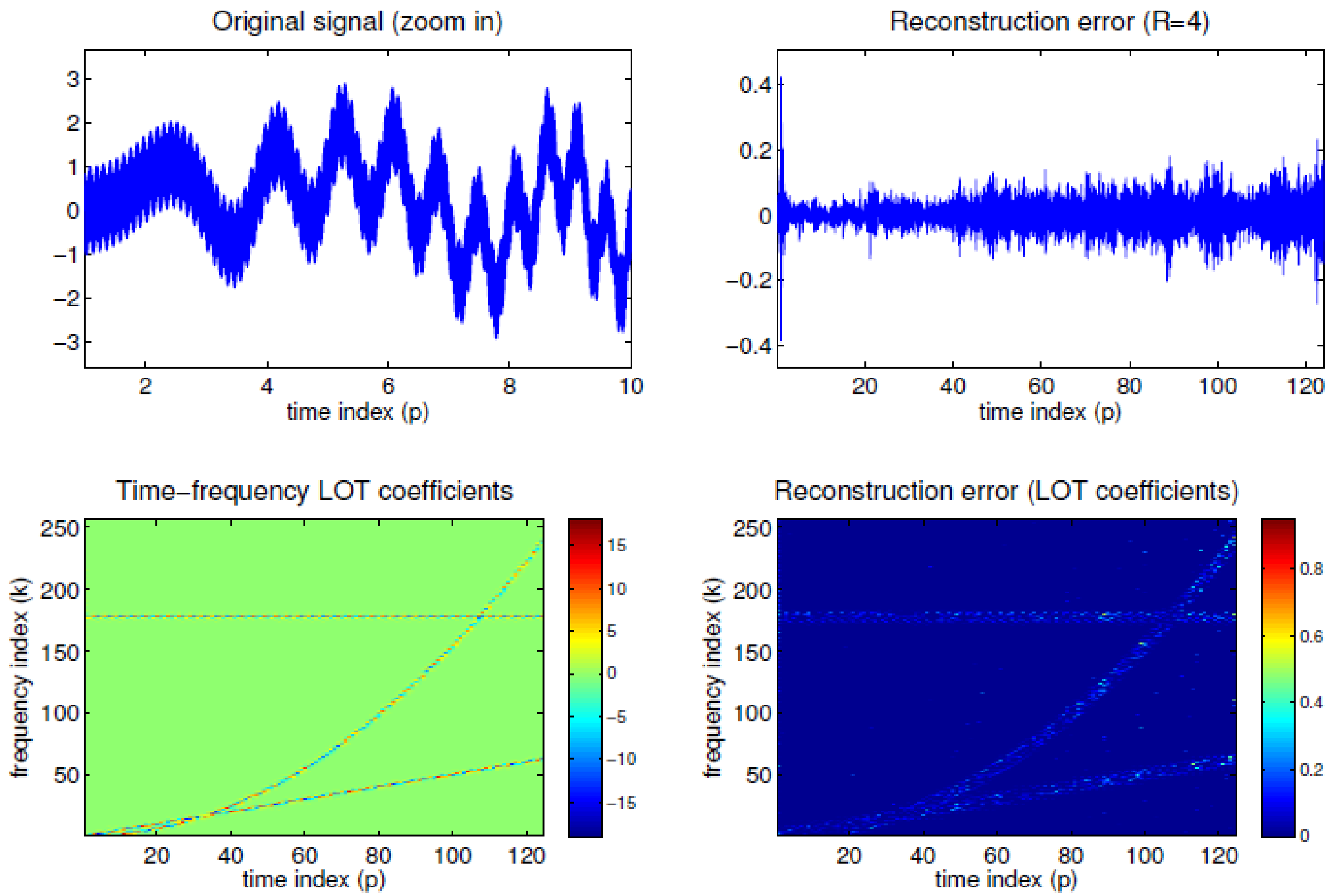
Streaming signal recovery - Results



(left) SER at different R from ±1 random measurements at 35 db SNR
(middle) Count for matrix-vector multiplications
(right) Matlab execution time

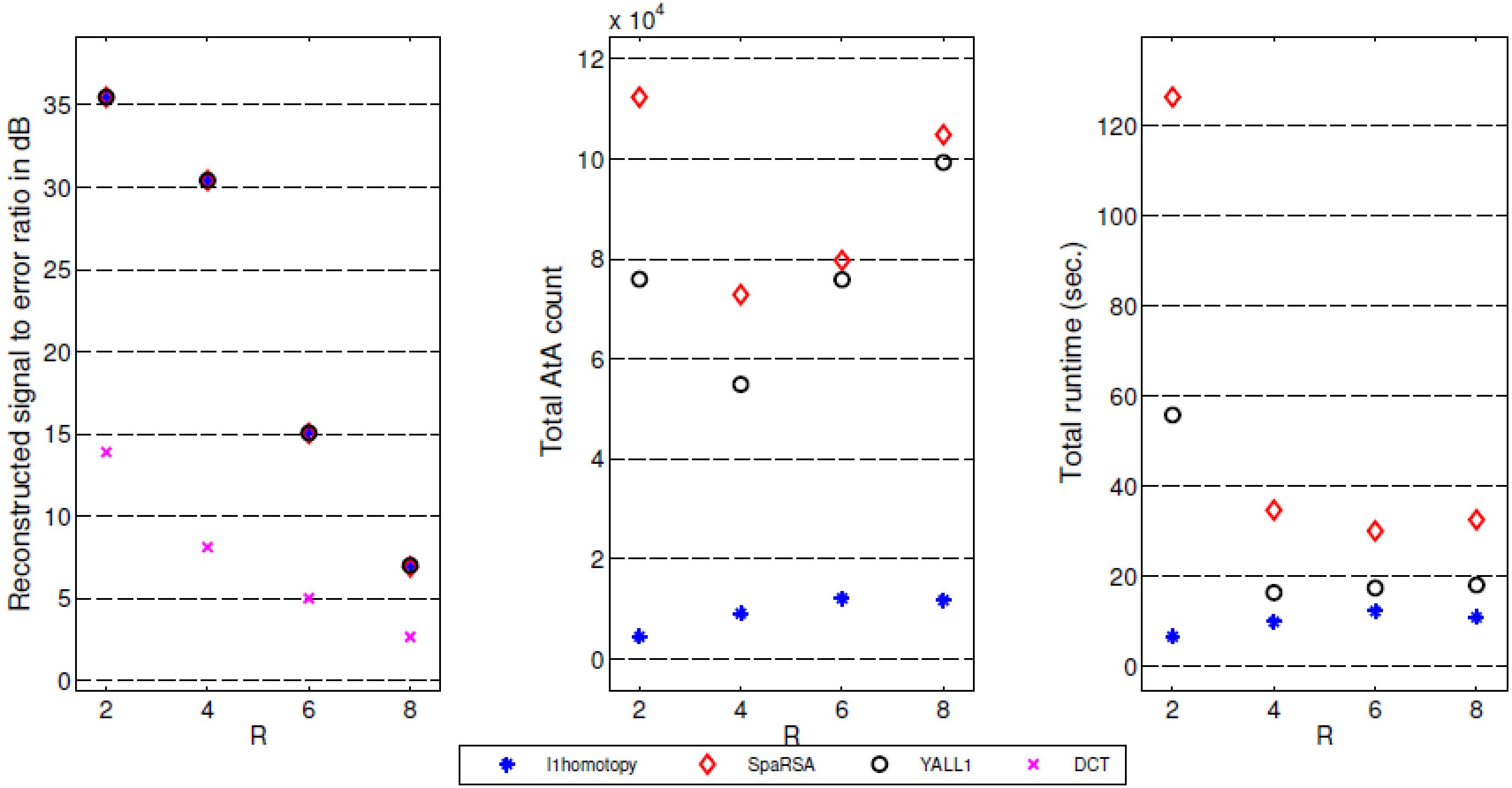
SpaRSA: Sparse Reconstruction by Separable Approximation, Wright et al., 2009. http://www.lx.it.pt/~mtf/SpaRSA/
YALL1: Alternating direction algorithms for L1-problems in compressive sensing, Yang et al., 2011. http://yall1.blogs.rice.edu/

Streaming signal recovery - Results



(Top-left) Mishmash signal (zoomed in for first 2560 samples.
(Top-right) Error in the reconstruction at R=N/M = 4.
(Bottom-left) LOT coefficients. (Bottom-right) Error in LOT coefficients

Streaming signal recovery - Results



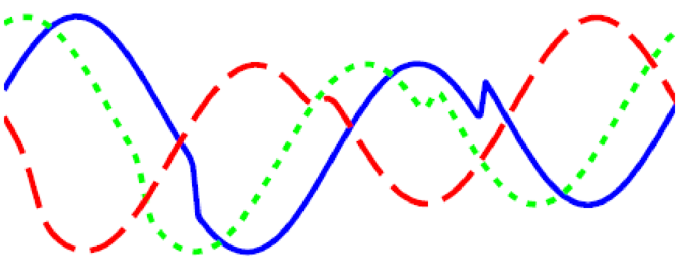
(left) SER at different R from ±1 random measurements at 35 db SNR
(middle) Count for matrix-vector multiplications
(right) Matlab execution time

Sparse recovery: dynamical system

- Signal observations: $y_t = \Phi_t x_t + e_t$
- Sparse representation: $x[n] = \sum_{p,k} \alpha_{p,k} \psi_{p,k}[n]$
- Linear dynamic model: $x_{t+1} = F_t x_t + f_t$

$$\underset{\alpha_1,\ldots,\alpha_P}{\text{minimize}} \sum_{t=1}^{P} \|W_t \alpha_t\|_1 + \frac{1}{2}\|\Phi_t \Psi_t \alpha_t - y_t\|_2^2 + \frac{\lambda_t}{2}\|F_{t-1}\Psi_{t-1}\alpha_{t-1} - \Psi_t \alpha_t\|_2^2$$

Dynamic model



Related work: [Vaswani ’08; Carmi et al. ’09; Angelosante et al. ’09; Zaniel et al. ’10; Charles et al. ’11]

Sparse recovery: dynamical system

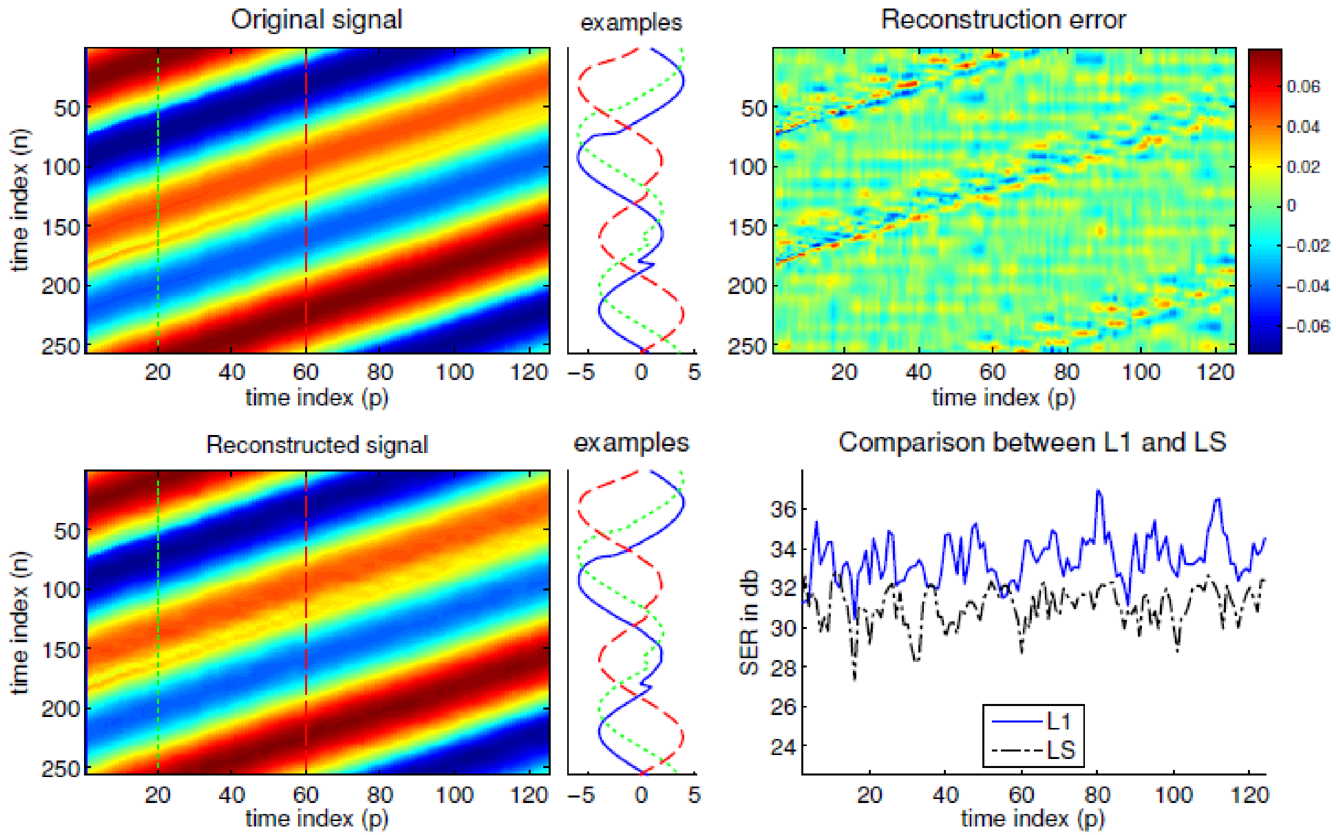
- Signal observations: $y_t = \Phi_t x_t + e_t$
- Linear dynamic model: $x_{t+1} = F_t x_t + f_t$
- Sparse representation: $x[n] = \sum_{p,k} \alpha_{p,k} \psi_{p,k}[n]$

$$\underset{\alpha_p}{\text{minimize}} \sum_{p=1}^{P} \|W_p \alpha_p\|_1 + \frac{1}{2}\|\Phi_p \Psi_p \alpha_p - y_p\|_2^2 + \frac{\lambda_p}{2}\|F_{p-1}\Psi_{p-1}\alpha_{p-1} - \Psi_p \alpha_p\|_2^2$$

$$\underset{\alpha}{\text{minimize}} \|\mathbf{W}\alpha\|_\mathbf{1} + \frac{\mathbf{1}}{\mathbf{2}}\|\bar{\mathbf{\Phi}}\bar{\mathbf{\Psi}}\alpha - \bar{\mathbf{y}}\|_\mathbf{2}^\mathbf{2} + \frac{\mathbf{1}}{\mathbf{2}}\|\bar{\mathbf{F}}\tilde{\mathbf{\Psi}}\alpha - \tilde{\mathbf{q}}\|_\mathbf{2}^\mathbf{2}$$

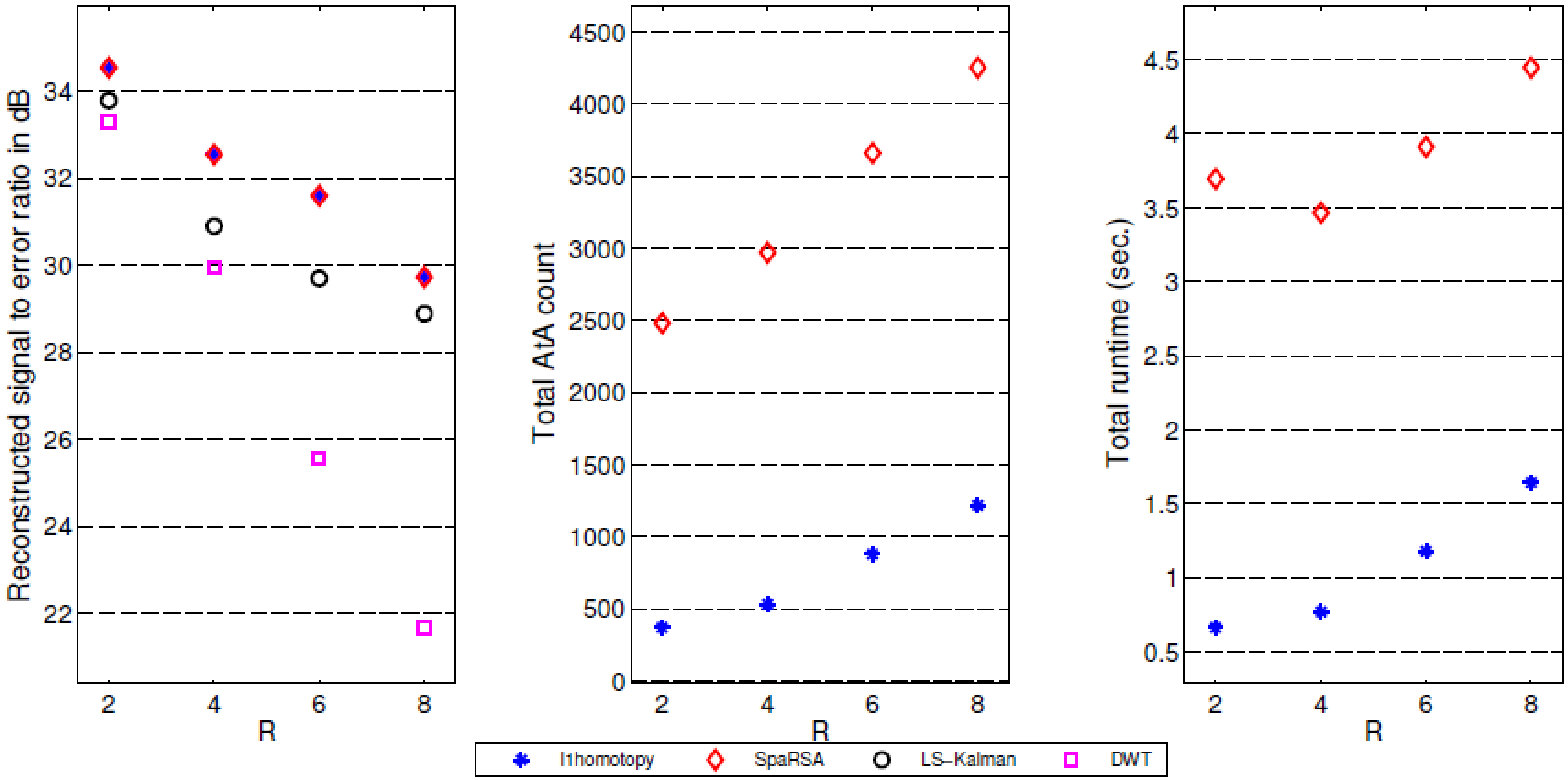
$$\text{minimize} \|\mathbf{W}\alpha\|_\mathbf{1} + \frac{\mathbf{1}}{\mathbf{2}}\|\bar{\mathbf{\Phi}}\tilde{\mathbf{\Psi}}\alpha - \tilde{\mathbf{y}}\|_\mathbf{2}^\mathbf{2} + \frac{\mathbf{1}}{\mathbf{2}}\|\bar{\mathbf{F}}\tilde{\mathbf{\Psi}}\alpha - \tilde{\mathbf{q}}\|_\mathbf{2}^\mathbf{2} + (\mathbf{1} - \epsilon)\mathbf{u}^\mathbf{T}\alpha$$

Dynamic signal recovery - Results



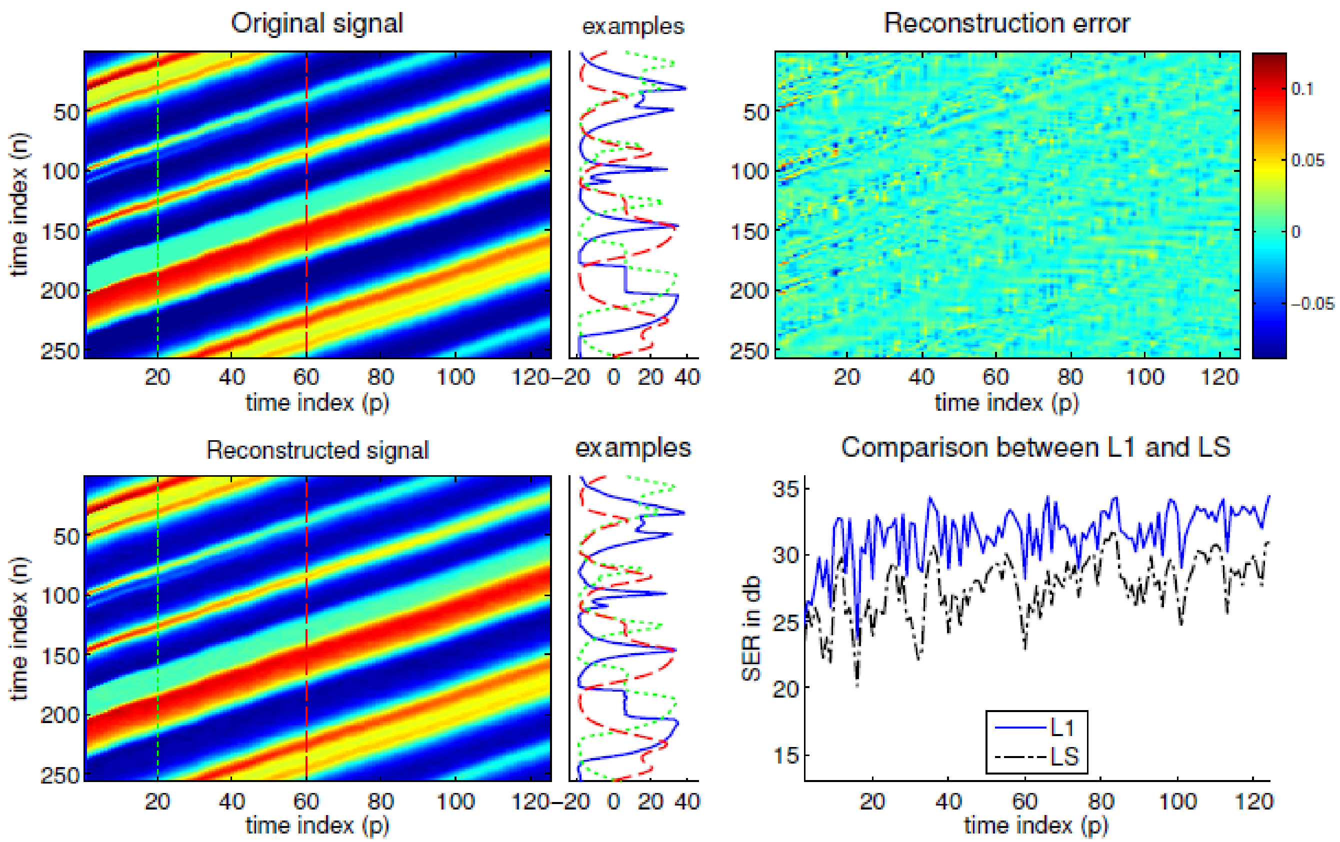
(Top-left) HeaviSine signal (shifted copies) in image
(Top-right) Error in the reconstruction at R=N/M = 4.
(Bottom-left) Reconstructed signal at R=4.
(Bottom-right) Comparison of SER for the L1- and the L2-regularized problems

Dynamic signal recovery - Results



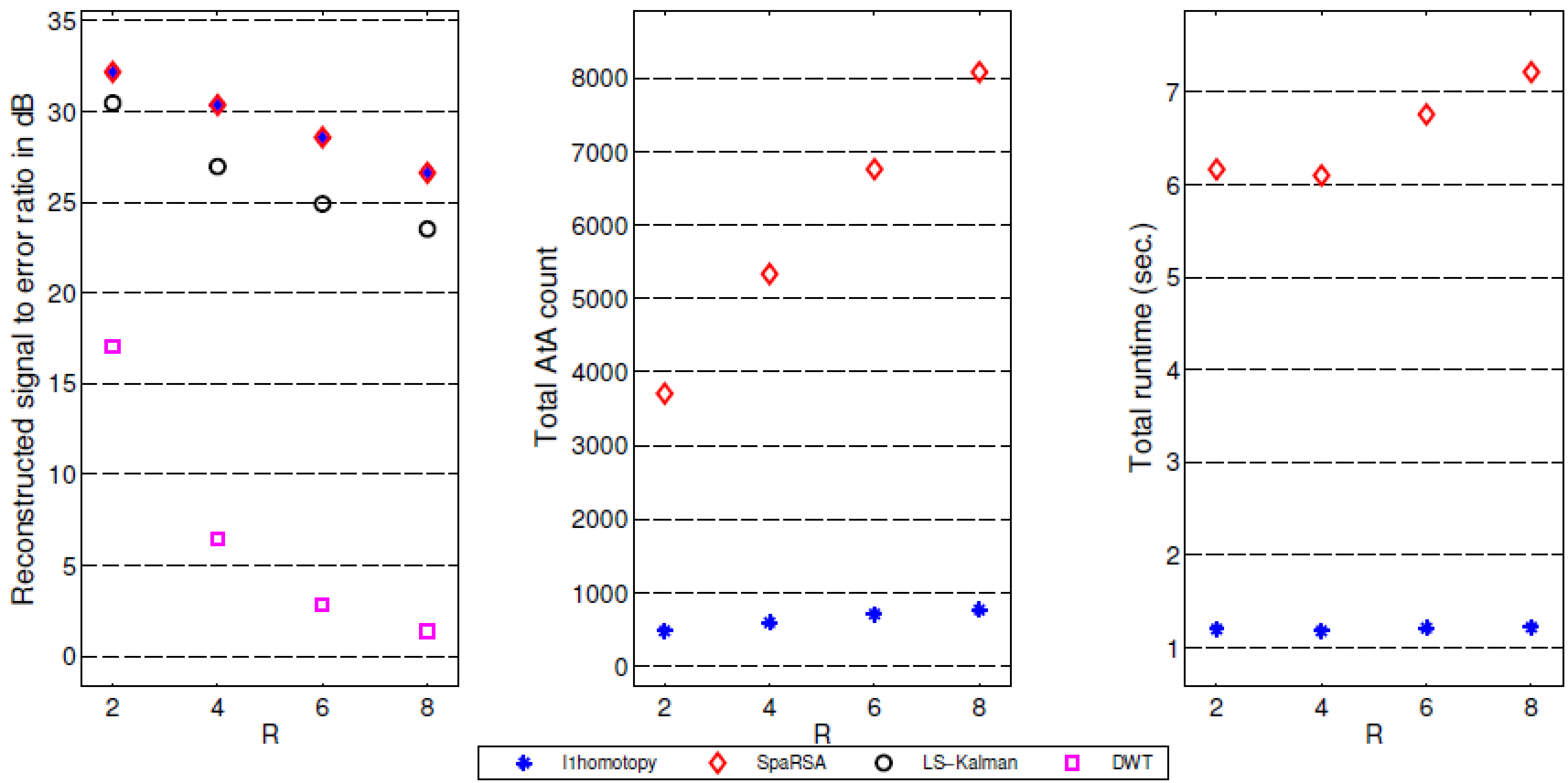
(left) SER at different R from ±1 random measurements at 35 db SNR
(middle) Count for matrix-vector multiplications
(right) Matlab execution time

Dynamic signal recovery - Results



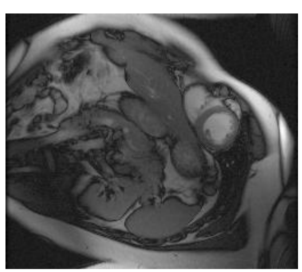
(Top-left) Piece-Regular signal (shifted copies) in image
(Top-right) Error in the reconstruction at R=N/M = 4.
(Bottom-left) Reconstructed signal at R=4.
(Bottom-right) Comparison of SER for the L1- and the L2-regularized problems

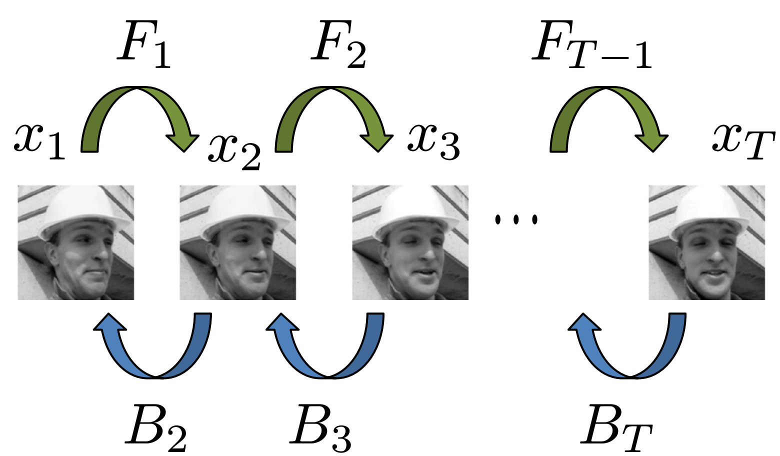
Dynamic signal recovery - Results



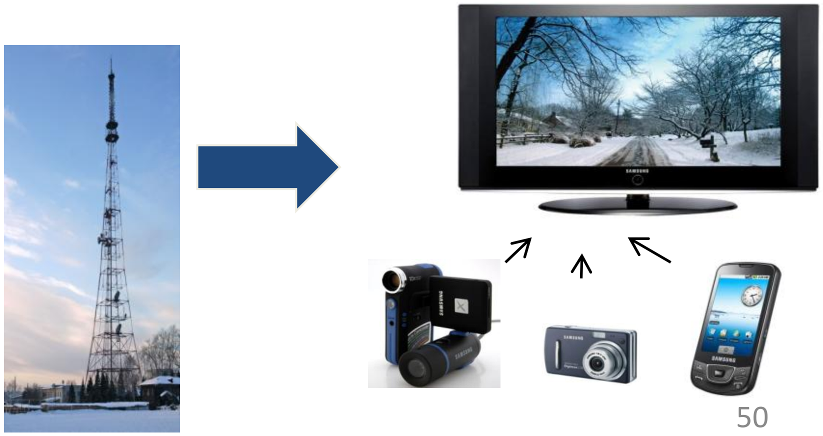
(left) SER at different R from ±1 random measurements at 35 db SNR
(middle) Count for matrix-vector multiplications
(right) Matlab execution time

Part 2: Dynamic models in video





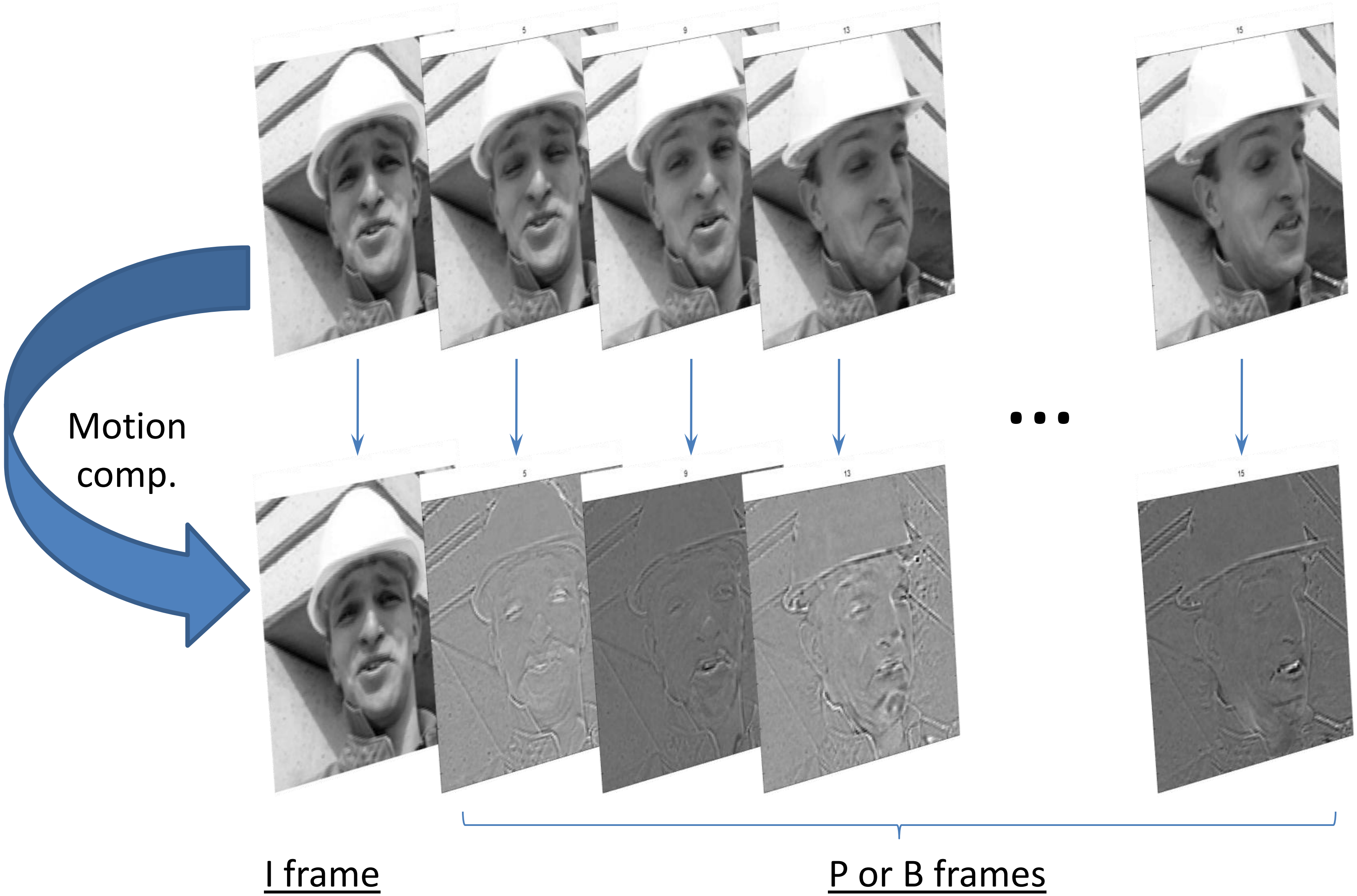
Low-complexity video compression



Video compression

Compression is achieved by removing the spatio-temporal redundancies in the videos

Video compression



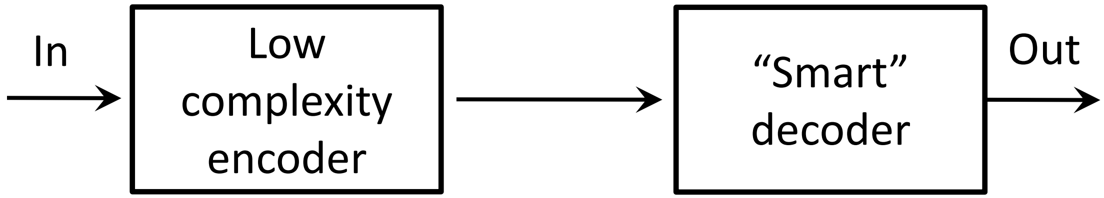
Compression is achieved by removing the spatio-temporal redundancies in the videos

Video coding paradigm

Major blocks in the encoder:

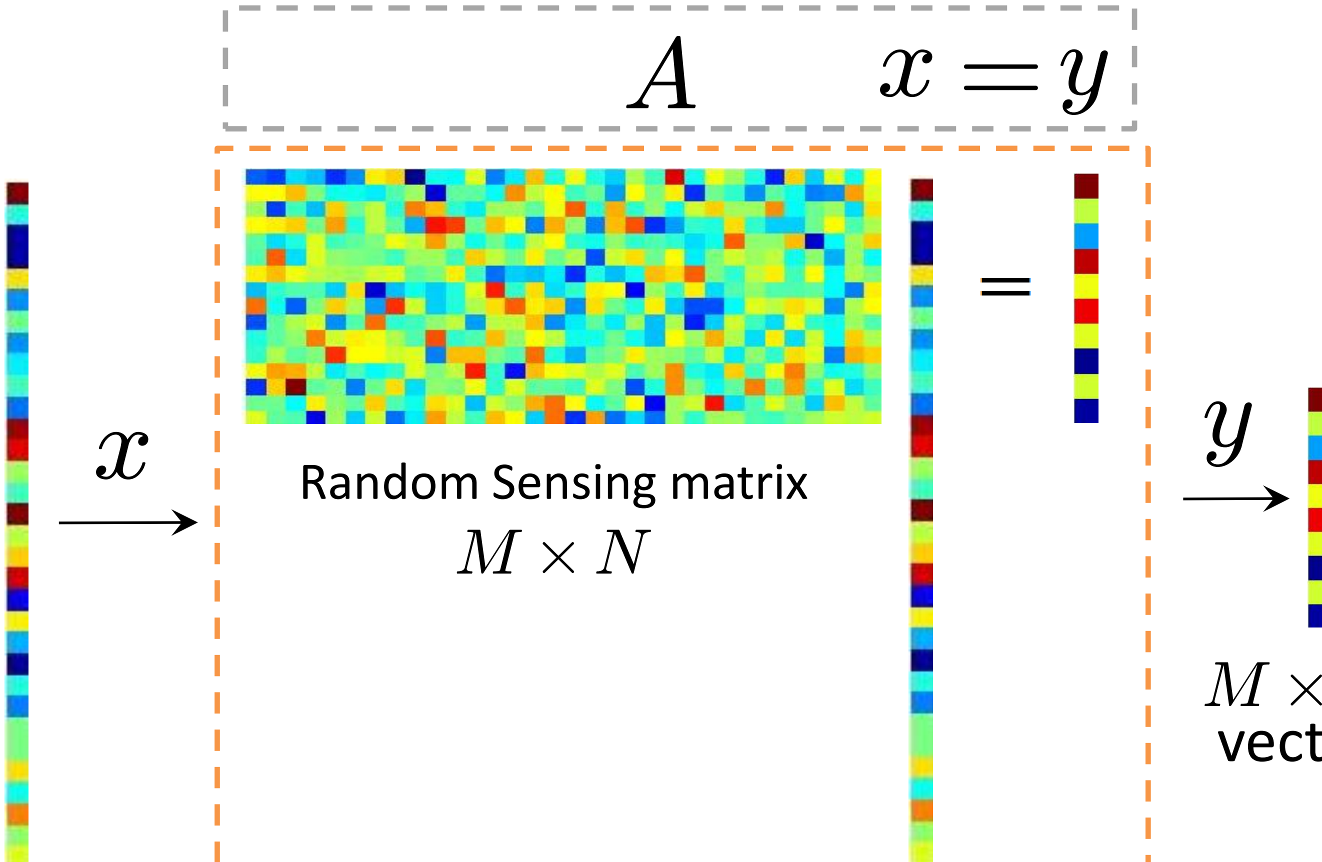
~~Motion estimation,~~ Transform coding, Entropy coding

Shift processing burden from the encoder to the decoder!



Compressive encoder

video frame
with N pixels



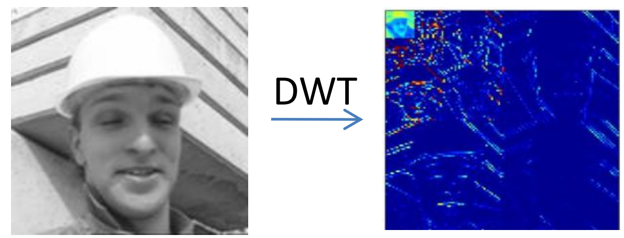
$N \times 1$
vector

$M \times 1$
vector

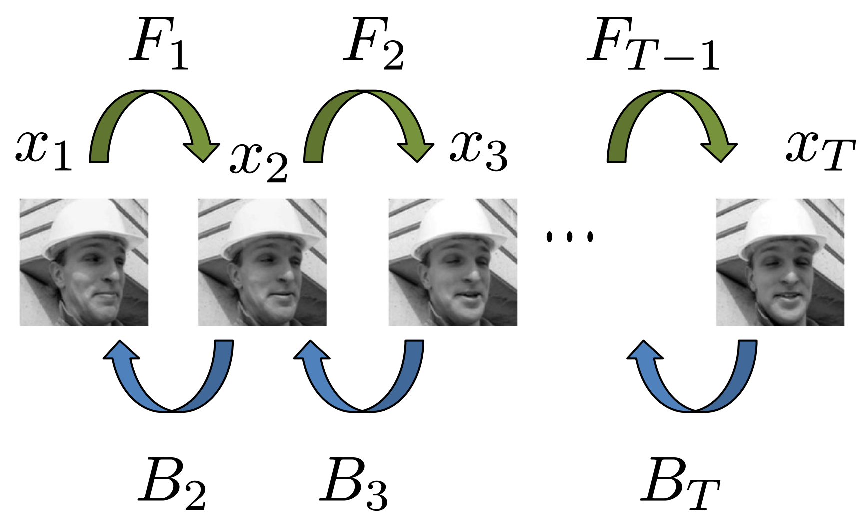
Encoder complexity is minimal, while all the computational load is shifted to the decoder

Structure in video (for recovery)

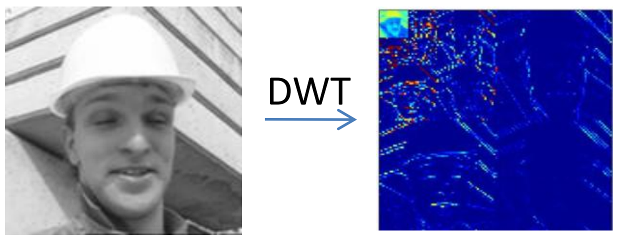
- Frame-by-frame recovery (same as recovery of independent images).
- Spatial structure:
  - Wavelets
  - Total-variation

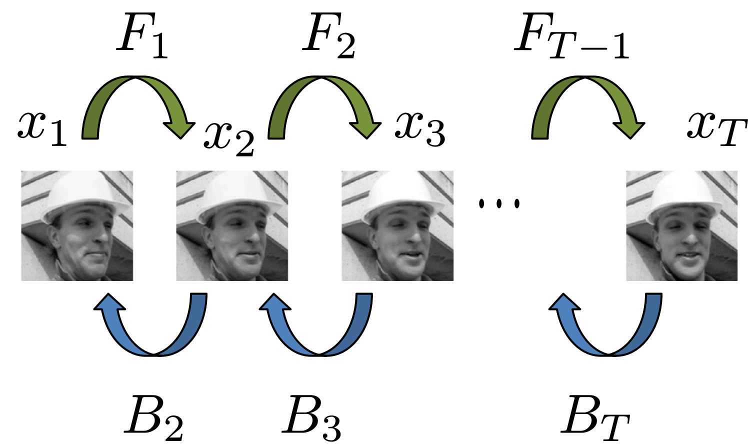


- Recovery of multiple frames (images are temporally correlated)
- Temporal structure:
  - Frame difference
  - Inter-frame motion



Structure...



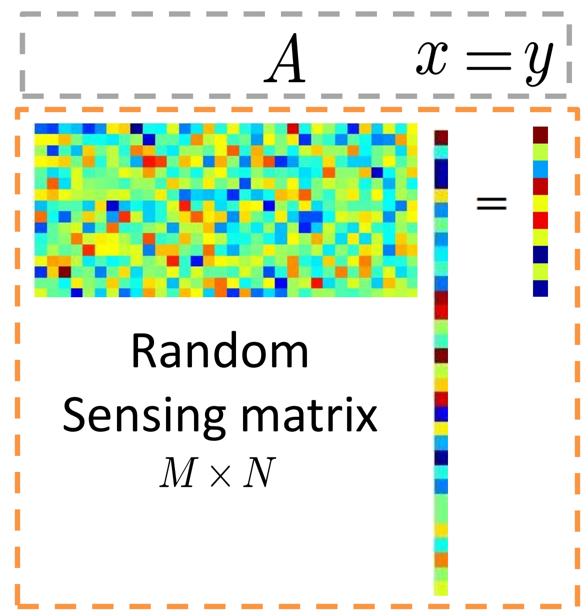


Linear dynamical system:

$$y_i = A_i x_i + e_i$$ (Linear measurements)

$$x_i = F_{i-1} x_{i-1} + f_i$$ (forward motion pred.)

$$x_i = B_{i+1} x_{i+1} + b_i$$ (backward motion pred.)



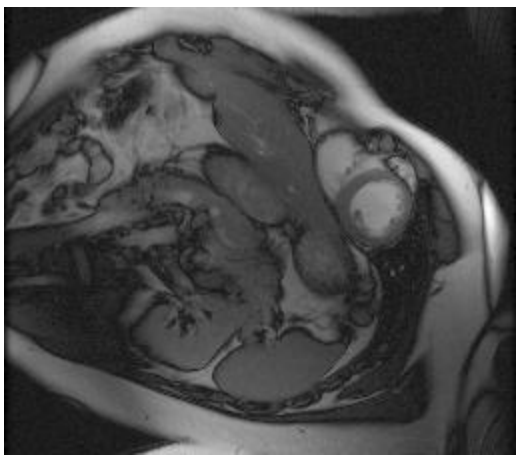
Accelerated dynamic MRI

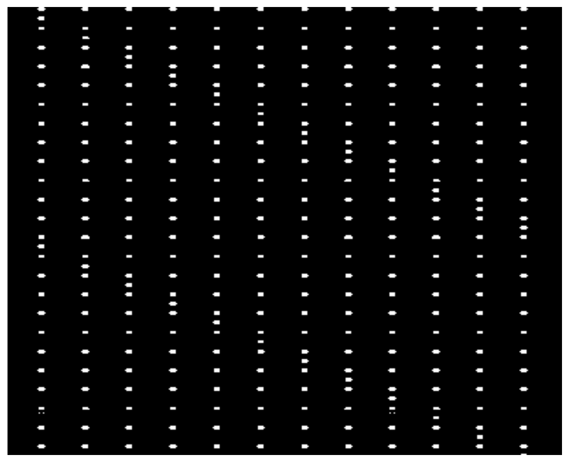
Accelerated acquisition in MRI?

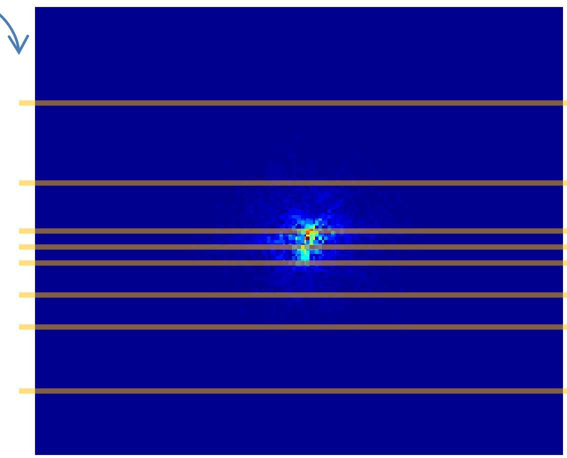
- How the cardiac MRI works:

A small spatial area is scanned in every heart beat (e.g., 8 lines per heartbeat)

One cardiac cycle







Temporal resolution

Fourier plane---Phase-encoding step samples rows in the Fourier plane

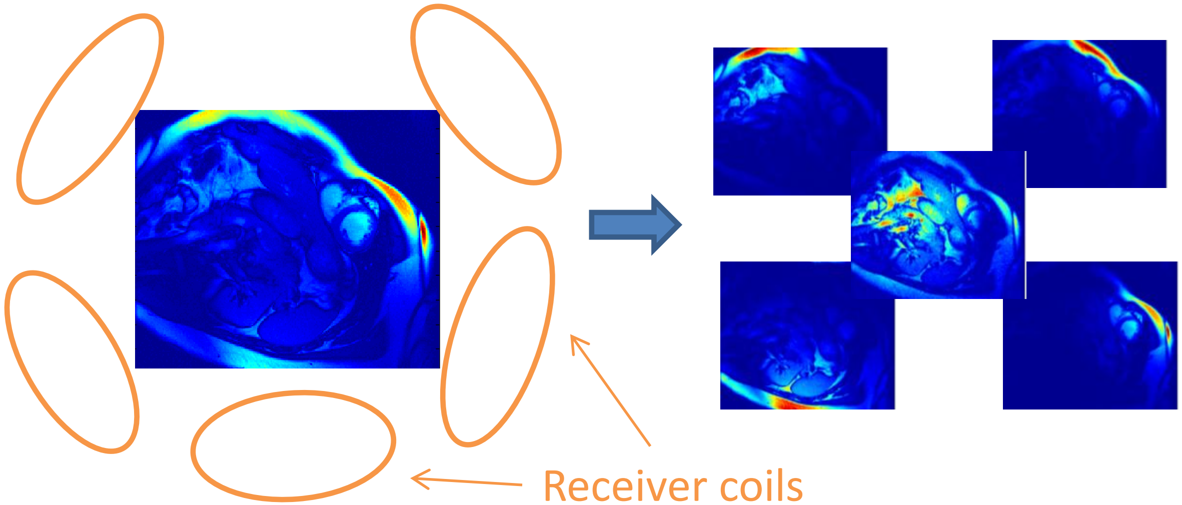
Acquisition during breathholds
Direct tradeoff between temporal resolution and lines per heart beat segment
Number of lines per frame are the same

Parallel imaging

- Parallel imaging [SENSE, SMASH, SPACE-RIP, GRAPPA, …]:

$$y_i = A_i x_i + e_i$$

$$A_i \equiv \begin{bmatrix} \mathcal{F}_{\ i} S_i^1 \\ \vdots \\ \mathcal{F}_{\ i} S_i^C \end{bmatrix}$$



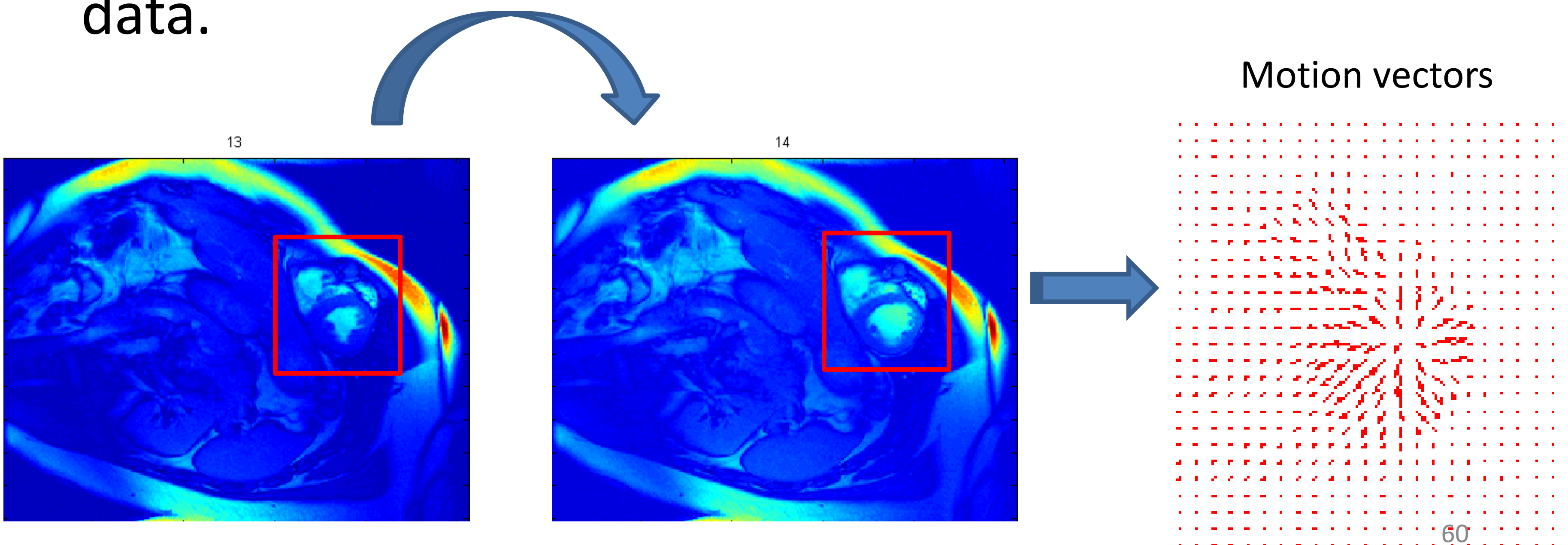
$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{bmatrix} = \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_T \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_T \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_T \end{bmatrix}$$

$$\equiv \mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{e}$$

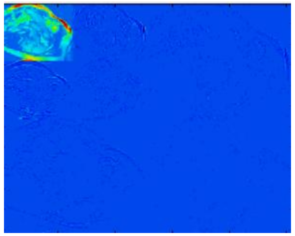
MRI data courtesy: Hamilton and Brummer

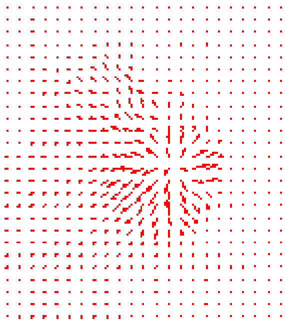
Motion-adaptive Spatio-temporal Regularization (MASTeR)

- We model temporal variations in the images using inter-frame motion!
- This way we are not using some fixed global model, instead, we learn the structure *directly* from the data.



Motion-adaptive Spatio-temporal Regularization (MASTeR)





- Spatial structure: wavelets
- Temporal structure: inter-frame motion
- Linear dynamical system model:

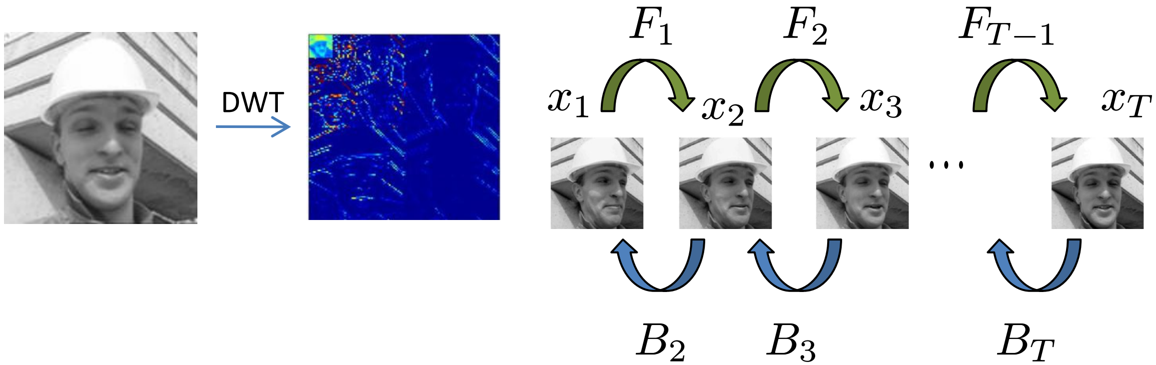
$$y_i = A_i x_i + e_i$$ (Linear measurements)

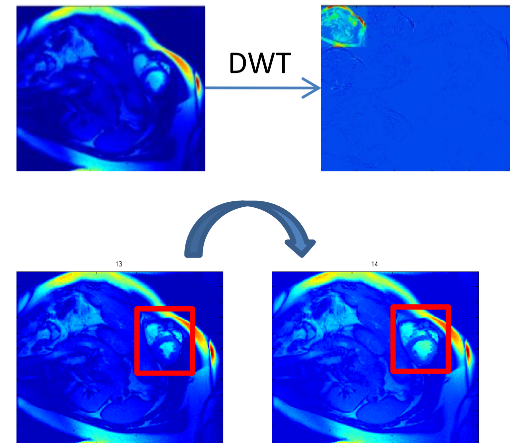
$$x_i = F_{i-1} x_{i-1} + f_i$$ (forward motion pred.)

$$x_i = B_{i+1} x_{i+1} + b_i$$ (backward motion pred.)

Video recovery

Video reconstruction



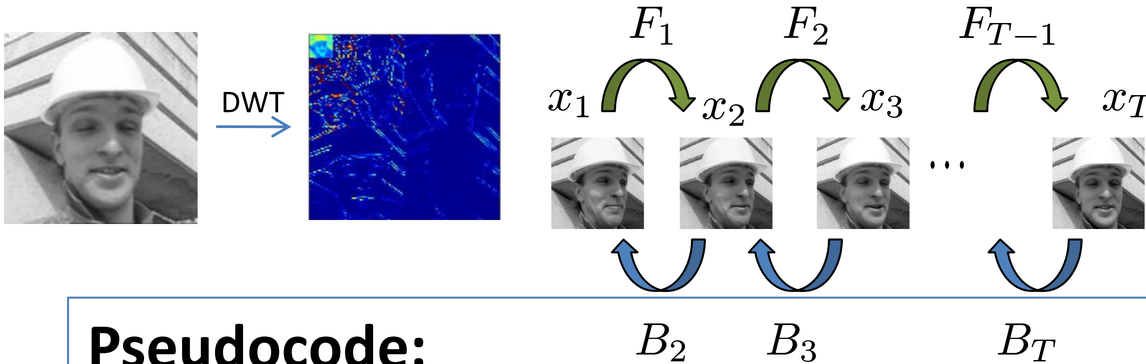


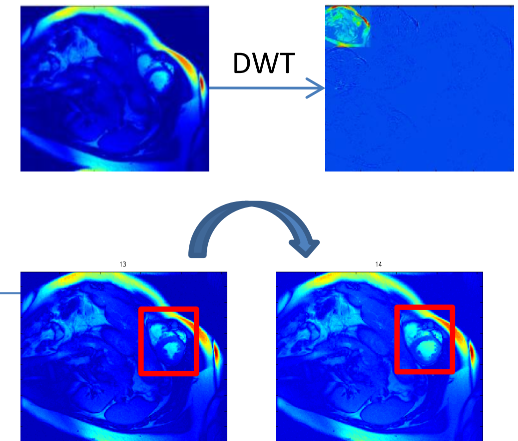
$y_i = A_i x_i + e_i$ (Linear measurements)

$x_i = F_{i-1} x_{i-1} + f_i$ (forward motion pred.)

$x_i = B_{i+1} x_{i+1} + b_i$ (backward motion pred.)

Video reconstruction





Pseudocode:

0. Find Initial frame estimates

Repeat

1. Calculate motion from frame estimates
2. Use the motion information to refine estimates

$$\underset{x_1,\ldots,x_T}{\text{minimize}} \sum_k \underbrace{\|A_k x_k - y_k\|_2}_{\text{Data fidelity}} + \underbrace{\tau_k \|x_k\|_{\Psi_1}}_{\text{Spatial regularity}} + \underbrace{\alpha_k \|F_k x_k - x_{k+1}\|_{\Psi_2}}_{\text{Forward motion diff.}} + \underbrace{\beta_k \|B_k x_k - x_{k-1}\|_{\Psi_3}}_{\text{Backward motion diff.}}$$

Comparison of low-complexity encoders

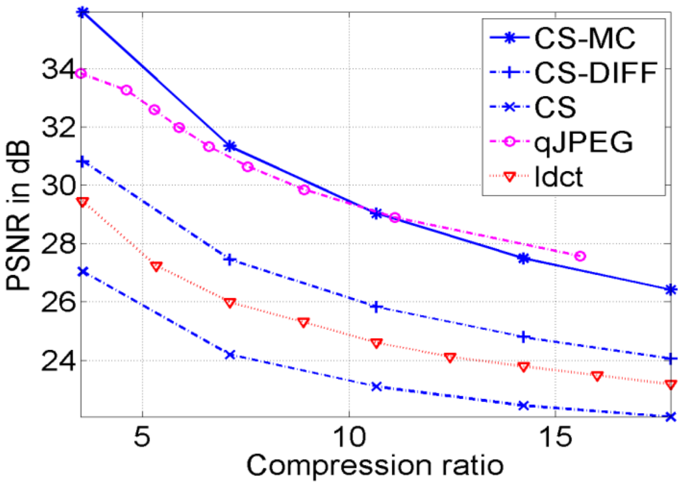
CS-MC: L1 with motion compensation
CS-DIFF: L1 with frame difference
CS: frame-by-frame

qJPEG: mask-based DCT thresholding
ldct: linear DCT (zigzag) thresholding

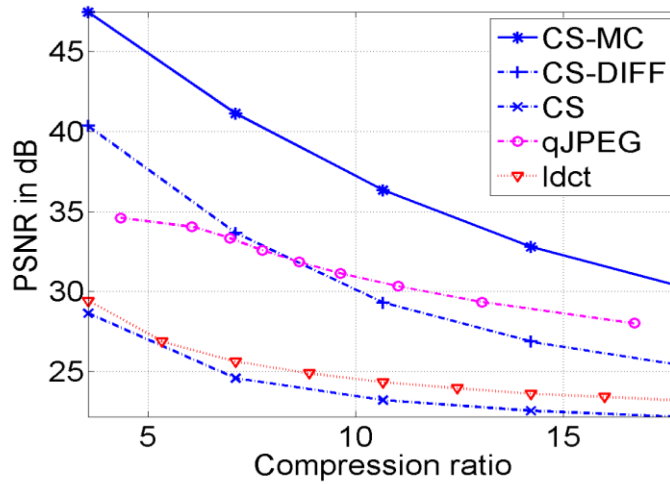
CS measurements: partial wavelets + noiselets
compression ratio = N/M


coastguard

coastguard



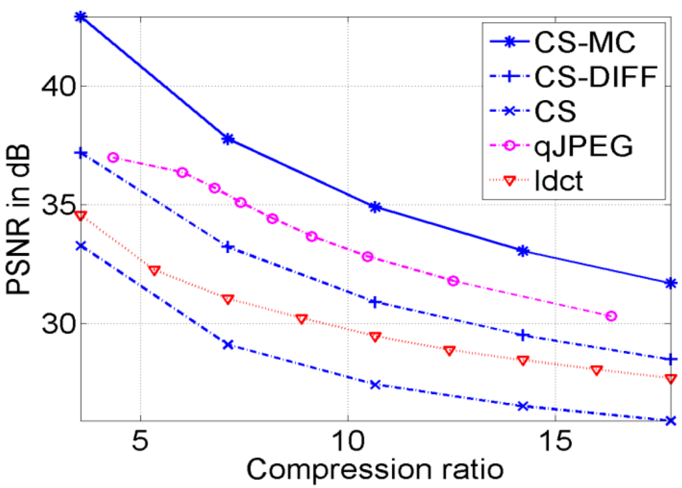
container



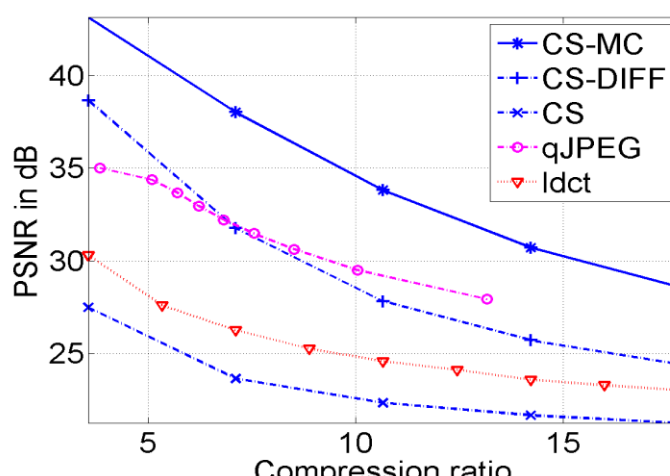

container


foreman

foreman



hall




hall

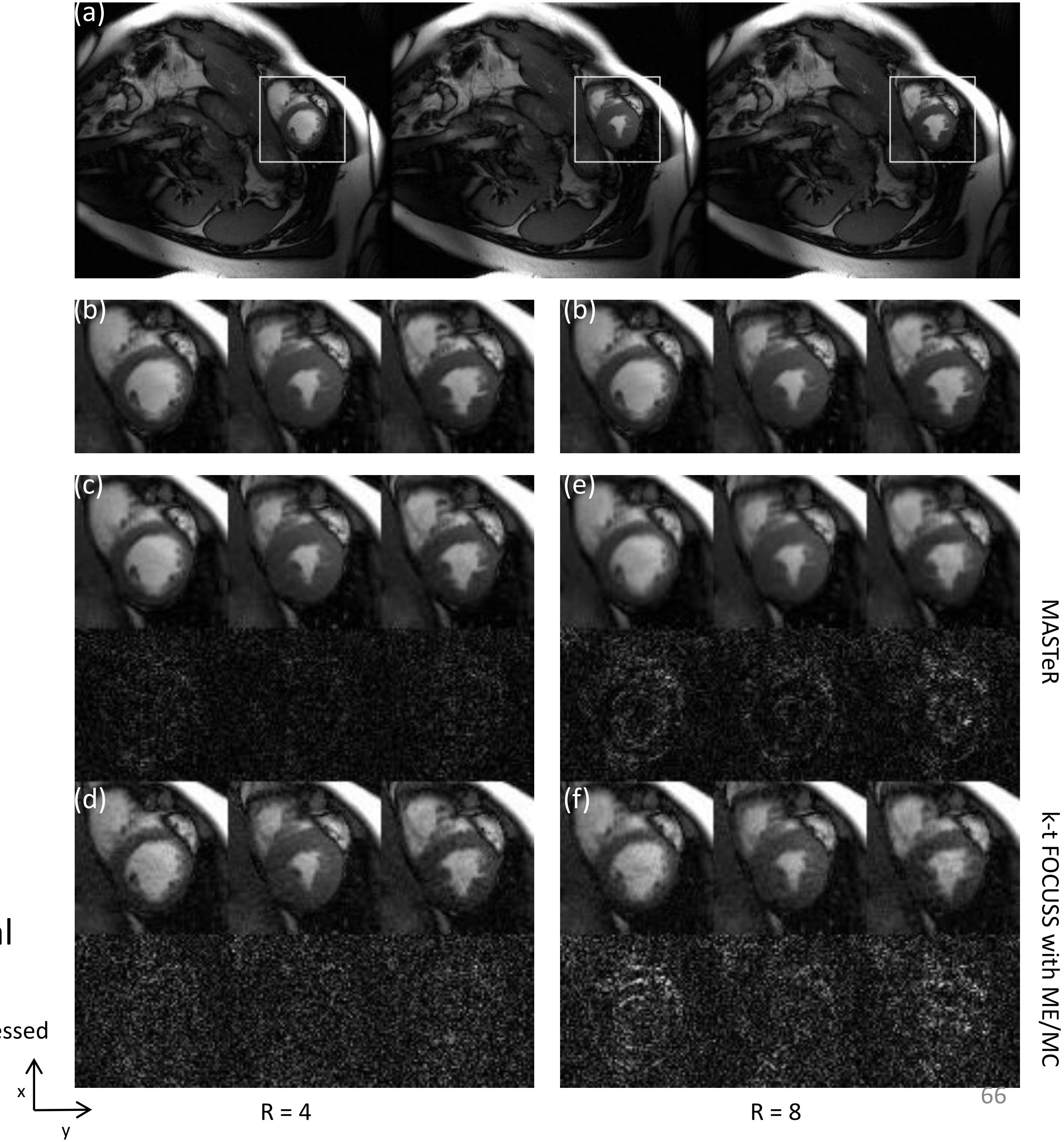
Results (short axis):

R: acceleration factor
Sampling: 8 low-freq. lines + random sampling

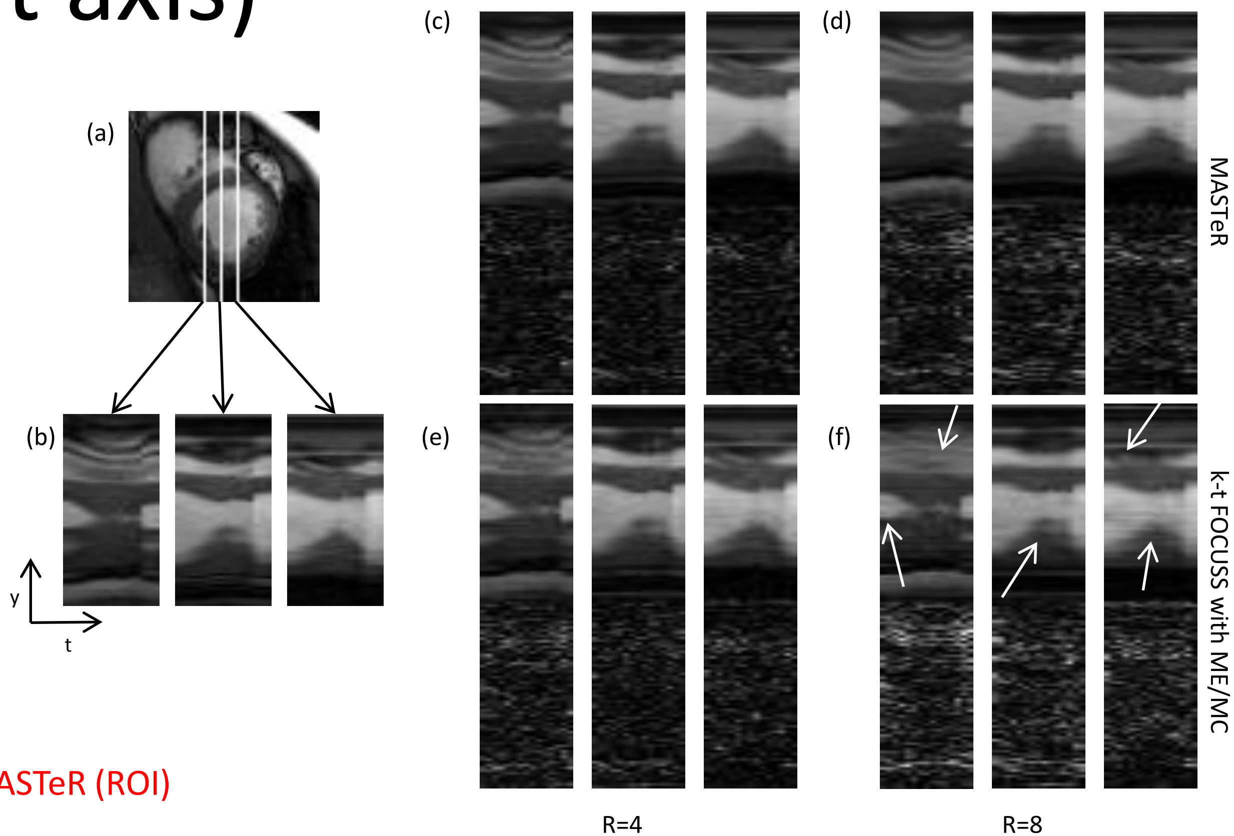
MASTeR: Motion-adaptive spatio-temporal regularization:

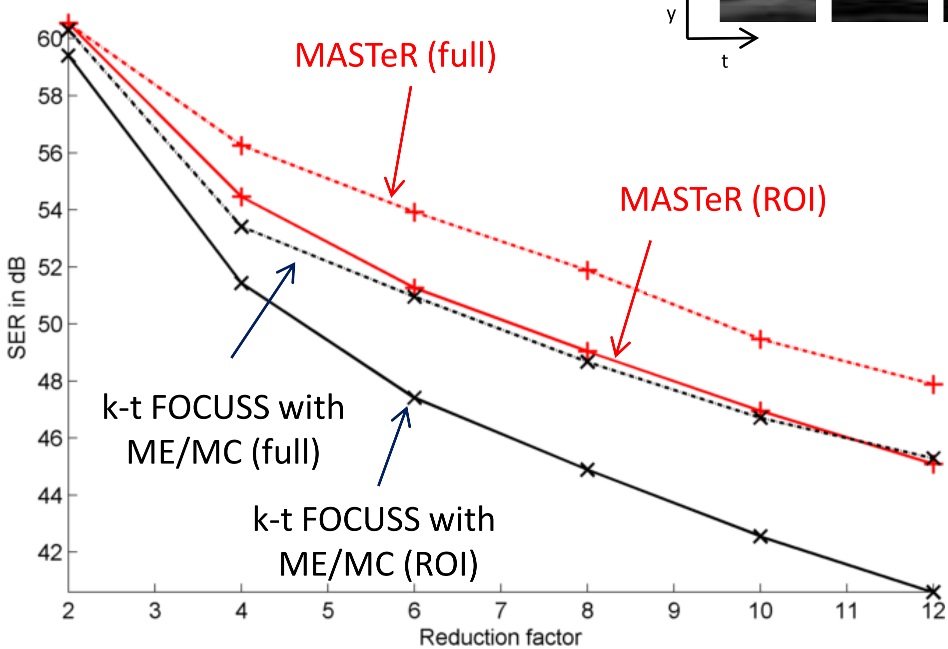
k-t FOCUSS with ME/MC:
-Temporal DFT sparsity
-Motion residual with a reference frame or temporal average.

[Jung et al., k-t FOCUSS: a general compressed sensing framework for high resolution dynamic MRI. MRM, 2009]



Results (short axis)





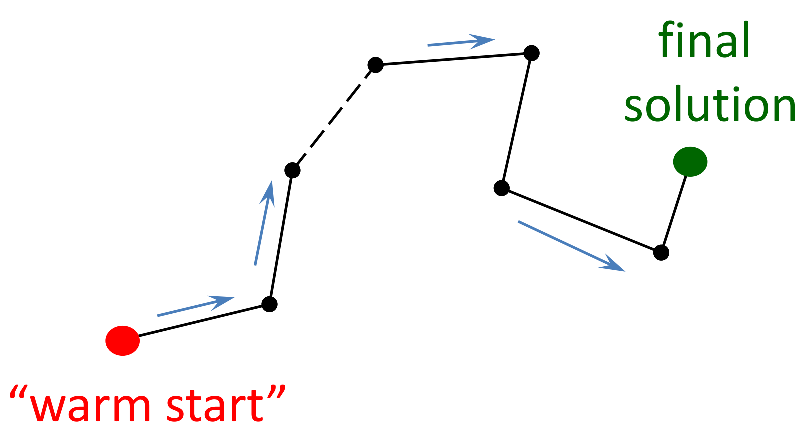
Conclusion

The more we know about the signal (dynamics)
the faster and/or more accurately we can reconstruct

Dynamic ℓ_1 updating

Quickly update the solution to accommodate changes

- ℓ_1 homotopy
  - Breaks updating into piecewise linear steps
  - Simple, inexpensive (rank-one update)

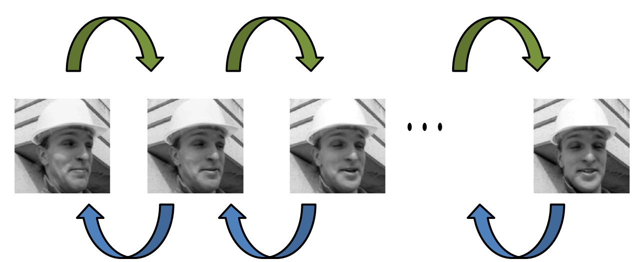


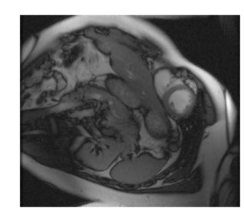
Dynamic modeling

Improve reconstruction by exploiting the dynamical signal structure

- Motion-adaptive dynamical system

1. Low-complexity video compression
2. Accelerated dynamic MRI





Future directions

- ℓ_1-homotopy:
  - Theoretical analysis
  - Large-scale streaming problems
- Compressive sensing in videos:
  - Adaptive sampling
  - Distributed cameras
  - Computational imaging (lightfield etc.)
- Medical imaging:
  - MRI… (maybe hyper-polarized)
  - Ultrasound
  - EEG

Select publications

- Dynamic updating
  - **M. Asif** and J. Romberg, ``Sparse recovery algorithm for streaming signals using L1-homotopy,'' Submitted to *IEEE Trans. Sig. Proc.*, June 2013. [Arxiv]
  - **M. Asif** and J. Romberg, ``Fast and accurate algorithm for re-weighted L1-norm minimization,'' Submitted to *IEEE Trans. Sig. Proc.*, July 2012. [Arxiv]
  - **M. Asif** and J. Romberg, ``Dynamic updating for L1 minimization,'' *IEEE Journal of Selected Topics in Signal Processing*, 4(2) pp. 421--434, April 2010.
  - **M. Asif** and J. Romberg, ``Sparse signal recovery and dynamic update of the underdetermined system,'' *Asilomar Conf. on Signals, Systems, and Computers*, Pacific Grove, November 2010.
  - **M. Asif** and Justin Romberg, ``Basis pursuit with sequential measurements and time-varying signals,'' in *Proc. Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP)*, Aruba, December 2009.
  - **M. Asif** and J. Romberg, ``Dynamic updating for sparse time-varying signals,'' *Conference on inf. sciences and systems (CISS)*, Baltimore, March 2009.
  - **M. Asif** and J. Romberg, ``Streaming measurements in compressive sensing: L1 filtering,'' *Asilomar Conf. on Signals, Systems, and Computers*, Pacific Grove, CA, October 2008.
- Dynamic model in video
  - **M. Asif**, L. Hamilton, M. Brummer, and J. Romberg, ``Motion-adaptive spatio-temporal regularization (MASTeR) for accelerated dynamic MRI,'' accepted in *Magnetic Resonance in Medicine*, November 2012 [Early view DOI: 10.1002/mrm.24524].
  - **M. Asif**, F. Fernandes, and J. Romberg, ''Low-complexity video compression and compressive sensing,'' preprint 2012.

Select publications

- Sparsity and dynamics
  - **M. Asif**, A. Charles, J. Romberg, and C. Rozell, ``Estimating and dynamic updating of time-varying signals with sparse variations,'' in *Proc. IEEE Int. Conf. on Acoustics, Speech, and Signal Processing (ICASSP)*, Prague, Czech Republic, May 2011.
  - A. Charles, **M. Asif**, J. Romberg, and C. Rozell, ``Sparse penalties in dynamical system estimation,'', *Conference on Inf. Sciences and Systems (CISS)*, Baltimore, Maryland, March 2011.
- Streaming greedy pursuit
  - P. Boufounos and **M. Asif**, ``Compressive sensing for streaming signals using the streaming greedy pursuit,'' in *Proc. Military Commun. Conf. (MILCOM)*, San Jose, California, October 2010.
  - **M. Asif**, D. Reddy, P. Boufounos, and A. Veeraraghavan, ``Streaming compressive sensing for high-speed periodic videos,'' in *Proc. IEEE Int. Conf. on Image Processing (ICIP)*, Hong Kong, September 2010.
  - P. Boufounos and **M. Asif**, ``Compressive sampling for streaming signals with sparse frequency content,'' *Conference on Inf. Sciences and Systems (CISS)*, Princeton, New Jersey, March 2010.
- Channel protection
  - **M. Asif**, W. Mantzel, and J. Romberg, ``Random channel coding and blind deconvolution,'' *Allerton Conf. on Communication, Control, and Computing*, Monticello, Illinois, October 2009.
  - **M. Asif**, W. Mantzel, and J. Romberg, ``Channel protection: Random coding meets sparse channels,'' *Information Theory Workshop*, Taormina, Italy, October 2009.

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Thank you!

Questions?

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